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MATHEMATICS MAGAZINE

JOHN ISBELL'S GAME

OF

BEANSTALK

AND

JOHN CONWAY'S GAME

OF

BEANS-DON'T-TALK

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SMITH NUMBERS • DIFFERENTIATION ARITHMETIC
TILING WITH CUBOIDS • PAIRS OF SPECIAL SQUARES

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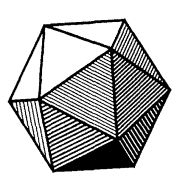

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The aim of *Mathematics Magazine* is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

In addition to articles and notes the *Magazine* solicits proofs without words, mathematical verse, anecdotes, cartoons, and other such material consistent with the level and aims described above. Letters and comments are also welcome.

The full statement of editorial policy appears in this *Magazine,* Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: G. L. Alexanderson, Editor, Mathematics Magazine, Santa Clara University, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Richard K. Guy ("John Isbell's Game of Beanstalk and John Conway's Game of Beans-Don't-Talk") is interested in number theory and all branches of combinatorics. He has edited the Unsolved Problems section of the *Monthly* since 1970, *Reviews in Number Theory, 1973–1983* for the A. M. S. and, currently, the *Guidelines* for the MAA committee, COMET. He has written *Unsolved Problems in Number Theory*, and is coauthor of *Winning Ways*, the definitive work on combinatorial games. Now being produced are *Fair Game*, a booklet for COMAP aimed at high school students; *The Book of Numbers*, a joint work with John Conway for Scientific American Library, *Unsolved Problems in Geometry* (with Ken Falconer & Hallard Croft) and a chapter on Combinatorial Games for the forthcoming *Handbook of Combinatorics*. In his seventies, he regularly accompanies his wife Louise into the mountains, on foot or on skis.

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John Isbell's Game of Beanstalk and John Conway's Game of Beans-Don't-Talk

RICHARD K. GUY
The University of Calgary
Alberta, Canada, T2N 1N4

John Isbell's game is played between Jack and the Giant. The Giant chooses a positive integer, say n_0. Then Jack and the Giant play alternately $n_1, n_2, n_3, \ldots$ according to the rule

$$n_{i+1} = \begin{cases} n_i/2 & \text{if } n_i \text{ is even,} \\ 3n_i \pm 1 & \text{if } n_i \text{ is odd,} \end{cases}$$

i.e., if n_i is even, there is only one option, while if n_i is odd, there are just two options. The winner is the player who moves to 1.

The rule looks suspiciously like the notorious Collatz problem [3], [4]. Is this game as hard to analyze?

In the first game, the Giant chose 1, so they outlawed that as a first choice. Then the Giant chose 64, so they banned powers of 4 as well. The Giant was smart enough not to choose an odd power of 2, and in the third game he chose 6 (Jack's automatic moves are in parentheses):

$$6 \quad (3) \quad 10 \quad (5) \quad 16 \quad (8) \quad 4 \quad (2) \quad 1$$

and the Giant won again.

In order to get Jack to play some more games, the Giant promised always to start with an odd number greater than 1. Then Jack was much happier, especially when he discovered

The Magic Beans Theorem

> If $n_0 > 1$ is odd,
>
> Jack need never lose.

Proof. If n_{2i} is odd, say of shape $4t \pm 1$, $t > 0$ (true for $i = 0$), then Jack has the option $n_{2i+1} = 12t \pm 2$ and the Giant must play $n_{2i+2} = 6t \pm 1$, which is odd and greater than $4t \pm 1$, so Jack can always prevent the Giant from descending to 1.

We'll call this Jack's **hiding strategy**, in which he stays in the beanstalk. Unfortunately, it grows to infinity. For example (Jack's moves are in parentheses again):

$$3 \quad (10) \quad 5 \quad (14) \quad 7 \quad (22) \quad 11 \quad (34) \quad 17 \quad (50) \quad 25 \quad (74) \quad 37\ldots$$

The Giant's moves,

$$3, 5, 7, 11, 17, 25, 37, 55, 83, 125, 187, 281, 421, 631, 947, 1421, \ldots$$

are like, but not quite the same as, the integer part of the powers of $3/2$, sequence 245 in Neil

Sloane's Handbook [**5**]:

$$1,2,3,5,7,11,17,25,38,57,86,129,194,291,437,656,985,1477,\ldots$$

Of course, if Jack is more adventurous, he may be able to win. For example, if the Giant chooses 3 or 5, then Jack wins just as the Giant did when the Giant chose 6. Can Jack always win when the Giant chooses an odd number bigger than 1?

Is there a Golden Eggs Strategy?

If so, in how few moves can Jack escape with the golden eggs?

In game four the Giant chose 7. Jack tried

The Greedy Strategy

Always descend when it's safe to do so.

Jack's two options are always of the form $2d$ and $2^g d$, where d is an arbitrary odd number (not the same one, each time it appears) and $g > 1$. If g is **odd**, it's safe to choose this option, because, after g moves, the first and last both being by the Giant, the number will be d (odd) and it's once again Jack's turn to play. But be careful!

Exception

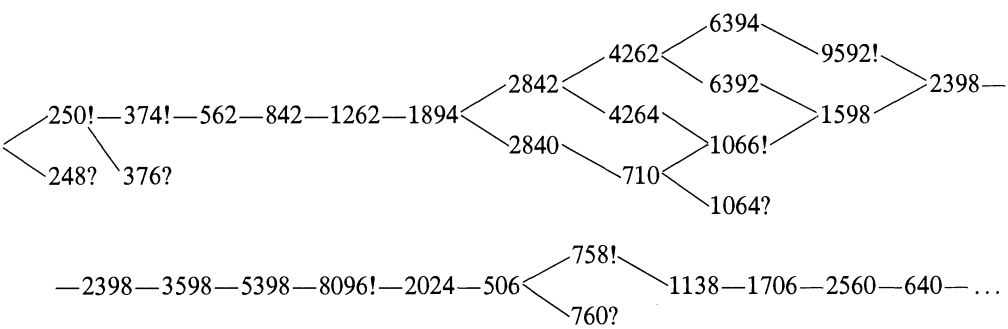
If $d = 1$, play 2^g just if g is **even**

How did game four go, when Jack tried the Greedy Strategy (underlined moves)?

7 (22) 11 (34) 17 (50) 25 (74) 37 (110) 55 (166) 83 (248) 124 (62) 31 (94) 47 (142) 71 (214) 107 (322) 161 (482) 241 (722) 361 (1082) 541 (1624) 812 (406) 203 (608) 304 (152) 76 (38) 19 (56) 28 (14) 7 ...

and he found himself back where he started!

Can Jack win from 7? He doesn't really have much option before the Giant plays 83. Now, with care, he can win. Only Jack's moves are shown; the Giant's are automatic.



and Jack wins in six more moves. Queries show places where the Greedy Strategy actually slows him down. *Festina lente*! Exclamation marks signify moves which are better than their alternatives. Of course, in order to win, some of these have to be greedy.

A Golden Eggs Strategy, if there is one, must involve **keeping control**, i.e., arranging that it is always Jack who has the two options. For, if Jack surrenders control, then the Giant could steal the putative Strategy and there would have been no point in Jack's stealing the golden eggs in the first place.

What expectation has Jack of winning? Let's make some assumptions. First, neglect the exception, $d = 1$. Powers of two are rare things. It's true, for Jack to win, he must get to a power of 2 eventually, but this only happens once, near the end of the game. Also assume that Jack uses the Greedy Strategy, even though we've seen an example where it doesn't do him any good. So the expectation may not be quite as good (how much worse?) as the one we're about to calculate.

We've seen that Jack's two options are $2d$ and $2^g d$, where d is odd (and we assume $d > 1$) and $g > 1$. We expect that $g = 2$ for half the time, $g = 3$ a quarter of the time, $g = 4$ an eighth of the time, and so on. Since the Greedy Strategy requires that g is odd, Jack will have to resort to the Hiding Strategy $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}$ of the time. So 2/3 of the time, a pair of moves, one by Jack and one by the Giant, will change the size of n in the ratio 3/2 (approximately).

$\frac{1}{4}$ of the time, 4 moves will change n in the ratio $\frac{3}{8}$,
$\frac{1}{16}$ of the time, 6 moves will change n in the ratio $\frac{3}{32}$,
$\frac{1}{64}$ of the time, 8 moves will change n in the ratio $\frac{3}{128}$,

and so on. So the expected change in ratio, in going from one odd number to the next, is

$$\left(\tfrac{3}{2}\right)^{2/3}\left(\tfrac{3}{8}\right)^{1/4}\left(\tfrac{3}{32}\right)^{1/16}\left(\tfrac{3}{128}\right)^{1/64}\cdots = \tfrac{3}{2^{17/9}} \approx 0.8100448,$$

and the number of moves this takes is expected to be

$$\left(\tfrac{2}{3}\times 2\right) + \left(\tfrac{1}{4}\times 4\right) + \left(\tfrac{1}{16}\times 6\right) + \left(\tfrac{1}{64}\times 8\right) + \cdots = \tfrac{26}{9}$$

Each move, on the average, changes n in the ratio

$$3^{9/26}\big/2^{17/26} \approx 0.9296726,$$

and we expect n to be halved about every $9\frac{1}{2}$ moves.

This is borne out well in practice. TABLE 1 gives the lengths of the shortest wins we have found for Jack, starting from each odd number less than 1000. The results are erratic, as might be expected, but if we average the entries in various neighborhoods, we see a general tendency to increase. For example, the averages of eight lengths for $n \pm 1, 3, 5, 7$, for various n, are

n	32	64	128	256	512	1024
average length	50	45.5	84	64	73.8	95.7
lb$n$	5	6	7	8	9	10
average/lb$n$	10	7.6	12	8	8.2	9.6

Here lb stands for binary log, to base 2. Our expectation is, that, starting with an odd $n_0 \approx 2^e$, the game will last about $9.5e$ moves. The average of the last line is, in fact, 9.2.

n_0	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49
	0	7	5	65	11	63	9	17	61	69	7	15	59	23	67	93	31	13	57	57	21	65	21	91	29
50 +	73	11	55	11	55	19	63	63	19	45	89	27	27	71	27	53	53	9	35	53	79	17	61	61	61
100 +	17	43	61	87	25	25	25	69	69	25	51	219	51	95	95	33	51	77	77	15	15	59	59	59	103
150 +	15	41	41	59	147	85	23	23	23	23	67	67	67	111	23	49	49	217	49	93	93	93	31	137	49
200 +	31	75	75	75	75	31	13	57	57	119	57	101	101	13	39	39	39	57	39	145	83	83	21	21	21
250 +	21	21	127	65	65	65	65	109	65	21	47	47	47	215	65	47	91	91	91	91	29	29	135	29	47
300 +	29	135	73	73	73	73	73	117	29	117	117	55	55	55	117	73	55	99	99	99	11	37	37	37	37
350 +	37	55	37	37	143	81	81	81	99	81	19	19	37	19	19	63	125	63	63	231	63	81	63	107	107
400 +	107	19	107	45	45	45	45	213	45	63	45	89	89	89	89	89	89	27	27	27	133	133	27	45	45
450 +	27	71	133	71	71	71	71	115	71	71	115	115	115	27	115	115	53	53	53	53	53	115	71	71	53
500 +	53	97	97	97	97	97	97	35	35	35	35	35	141	35	53	53	35	35	79	141	79	79	79	79	79
550 +	97	79	17	79	123	123	35	17	35	17	61	61	123	61	61	61	229	123	61	79	61	61	105	105	105
600 +	105	105	17	105	105	43	43	43	43	149	43	211	43	61	61	43	61	87	87	87	87	87	87	87	87
650 +	25	25	87	87	25	131	25	131	43	25	43	43	25	69	69	69	131	69	69	69	69	69	69	113	87
700 +	69	69	69	113	113	69	113	25	25	113	113	51	69	51	113	51	51	51	51	113	51	51	69	263	51
750 +	51	95	95	95	95	95	95	95	95	95	33	33	33	33	33	33	139	139	139	51	33	51	51	51	33
800 +	33	33	77	77	139	77	77	77	77	77	77	77	77	95	77	77	77	77	121	121	121	77	33	121	33
850 +	33	15	121	59	77	59	121	59	59	59	59	59	227	121	77	59	77	59	59	59	59	103	103	103	103
900 +	103	103	103	15	15	103	103	41	41	41	41	41	41	147	147	41	147	209	41	147	59	59	59	41	59
950 +	41	85	85	147	85	85	85	191	85	85	85	85	41	85	41	23	85	85	85	23	129	129	23	129	85
1000 +	41	41	23	41	41	23	67	67	67	67	67	129	235	67	67	67	67	67	235	129	67	67	111	67	85

TABLE 1. **Tentative list of least lengths, $l(n_0)$, of win in Beanstalk.**

We computed TABLE 1 by hand. It would be better to use a computer, though some care in programming is needed if you want to extend the results substantially, since a binary tree search is involved. In practice our approach was somewhat intuitive, but in theory you can ask a machine to use the Greedy Strategy, flagging each occasion that the greedy option was adopted. The machine continues until either

(a) it reaches 1, so that a win has been found, in l moves, say, where l is not necessarily the least possible, or

(b) it cycles (as it will if you start from $7, 11, 17, 19, 25, 31, 37, 47, 55, \ldots$) or

(c) it overflows.

In case (a), backtrack and pursue each nongreedy alternative, until either

(aa) a shorter win is found, and l can be updated, or

(ab) it cycles, or

(ac) m moves have been made and the number is greater than 2^{l-m}, so that no shorter win is possible. When the backtrack is complete, you know you have found the shortest win.

In case (b), backtrack similarly, and again, either

(ba) a win is found, or

(bb) it cycles again, or

(bc) it overflows.

In case (c) we should still backtrack. In practice, wins always seem to appear. The trouble is, we don't know if they're the shortest, but at least there's the incentive to increase the word length and try to complete the search.

Verification and extension of Table 1 would be welcome; or corrections, hopefully documented with the sets of options where the Greedy Strategy was *not* used. Here is a list of such options, $3n \pm 1$, of shape $2d$, d odd, that we found. This is naturally incomplete, but it may not even contain all examples less than, say, 1000.

134	374	994	1910	3398	6014	7646	14506	64262	166934	957706
202	542	1066	2218	3638	7018	8630	16906	78250	250402	
226	758	1478	2266	4234	7522	10526	28406	96394	275002	
250	898	1546	3062	5098	7526	11270	32798	102946	412502	

There are many places where the Hiding Strategy and the Greedy Strategy lead to the same shortest length of win. For example, 2840 or 2842 on the 25th move in the game (that Jack won) starting with 7, both lead to a win in 40 more moves, provided the best play is followed in each case. We haven't intentionally included any such examples in the above list, but it's hard to be sure.

In TABLE 1, as soon as we get away from the influence of the Strong Law of Small Numbers **[2]** we notice runs of consecutive odd numbers, n_0, which lead to the same least length of play, $l(n_0)$. When the length does change, it almost always does so by one of the numbers

$$18, 26, 44, 62, 88, 106, 150, 168, 212, \ldots$$

What are these differences? They are even numbers, representing several pairs of moves by Jack and the Giant, and are of the form $p+q$ where p/q is a good approximation to the binary log of 3,

$$\mathrm{lb}\,3 = \frac{\ln 3}{\ln 2} = 1.5849625\cdots = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{23+}\cdots$$

which has convergents

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}, \frac{485}{306}, \frac{1054}{665}, \frac{24727}{15601}, \ldots$$

which, with some "mediants" e.g., $11/7$ and $27/17$, account for most of the numbers, or their halves, in the above list. The rest correspond to other good approximations, such as $46/29$ & $103/65$.

For example, from $n_0 = 575$ we can get to 593 in $62 = 2 \times 31 = 2(19+12)$ moves. The Giant's 31 automatic halving moves are usually to odd numbers, but he makes seven even moves, e.g., 4372, 8300, 7004, 3940, 11216, 2804, and 1052, when Jack uses the Greedy Strategy. So there are $31+7$ "dividing by 2" moves and $31-7$ "multiplying by about 3" moves, and over the whole run of 62 moves we have multiplied by approximately

$$3^{24}/2^{38} = (3^{12}/2^{19})^2 = (1.013643\ldots)^2,$$

which is quite a good approximation to 1. In fact,

$$575 \times 3^{24}/2^{38} = 590.769\ldots$$

whose difference from 593 is accounted for by the plus and minus ones.

Rather than calculate the length of the game, Isbell found it easier to find the least k so that n_{2k} is either (a) an odd power of 2, or (b) an odd number smaller than the n_0 we started from. The first few values of this $k(n_0)$ are

n_0	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
$k(n_0)$	2	1	30	3	29	2	4	28	2	1	3	17	4	2	12	8	2	4	9	3	28

45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	81	83	85	87	89
2	11	4	2	3	3	2	8	2	10	27	2	8	10	3	3	2	4	5	2	1	3	7

91	93	95	97	99	101	103	105	107	109	111	113	115	117	119	121	123	125	127
4	2	9	26	2	4	7	3	9	2	7	4	2	3	3	4	97	2	8

129	131	133	135	137	139	141	143	145	147	149	151	153	155	157	159	161
12	2	6	4	3	3	2	4	22	2	26	3	11	4	2	45	8

163	165	167	169	171	173	175	177	179	181	183	185	187	189	191	193	195	197	199
2	4	6	3	20	2	5	4	2	3	3	96	25	2	7	8	2	24	5

Note that k may be much less than half the length of the shortest game; for example $k(19) = 2$. Moreover, the play that achieves k may not be the way to go to achieve the shortest win; for example, $n_0 = 45, 67, 75, 83, 113, 125, \ldots$

Isbell calls n_0 a **knot** if $k(n_0)$ exceeds all earlier k. The first three knots are $3, 7, 123$. What is the fourth? A candidate is 747, although $k(747) \leqslant 74$, since we can get to 473 in 148 moves, so this is ruled out if our calculation $k(123) = 97$ is correct.

How often does play cycle? Perhaps only for finitely many numbers? You can always get out of a cycle, by judicious use of the Hiding Strategy.

Are there odd numbers from which Jack can't win?

This seems very unlikely, but to prove that it never happens may be a very hard problem.

After I'd written this article I met John Conway. We started talking about Beanstalk, but it soon emerged that we were talking about games that were similar, but not the same. He disappeared and came back an hour later with two pages of typing, headed

BEANS DON'T TALK

John Isn'tbell

He had misremembered the rules, and arrived at another interesting game. The typical position is a whole number n, and there are two possible moves, from n to

$$(3n \pm 1)/2^*$$

where we write 2^* for the highest power of 2 that divides the numerator. As in Beanstalk, the winner is the person who moves to 1.

Are there any drawn positions?

You might naïvely answer "yes", and exhibit the perpetual play of Figure 1.

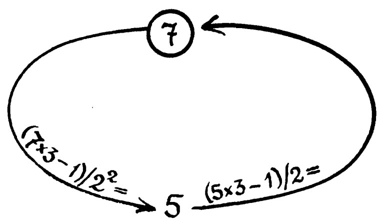


FIGURE 1. **A boring game of Beans-Don't-Talk. (A less trivial example is a modification of Jack's fourth game of Beanstalk, when he tried the Greedy Strategy.)**

But this is a drawn **play** of the game; not an example of a drawn **position**. The number 5 is an $\mathscr{N}$-**position**, or next-player-winning position, because she can play $(5 \times 3 + 1)/2^4 = 1$ and win immediately. On the other hand, 7 is a $\mathscr{P}$-**position** or previous-player-winning position, because, even if the present player chooses his other option $(7 \times 3 + 1)/2 = 11$, his opponent still wins immediately by playing $(11 \times 3 - 1)/2^5 = 1$. What we'd like to know is:

> Are there any $\mathscr{O}$-positions?

That is, **open positions**, neither $\mathscr{N}$-positions nor $\mathscr{P}$-positions, from which neither player can force a win against the best defence.

David Seal used a microcomputer to show that 2899 is the smallest number whose analysis involves numbers greater than 20,000. All earlier numbers are $\mathscr{N}$-positions or $\mathscr{P}$-positions.

Impartial, last-player-winning games of this kind are conveniently analyzed using a function first introduced by Steinhaus [7], which Cedric Smith [6] appropriately calls the **remoteness** of the position [see **1**, Chap. 9]. In Beans-Don't-Talk, the remoteness of 1 is zero, because the game is over. The remoteness of 5 and of 11 is one, because the game can be won in one move. Intuitively, the remoteness is the number of moves that the game will last if the winner is trying to win as quickly as possible, and the loser is trying to postpone defeat as long as possible.

> Positions of odd remoteness are $\mathscr{N}$-positions.
> Positions of even remoteness are $\mathscr{P}$-positions.

The $\mathscr{O}$-positions have infinite remoteness.

> Does Beans-Don't-Talk have positions of infinite remoteness?

FIGURE 2 shows plays starting from the $\mathscr{P}$-position 23. The positions are arranged according to their remoteness and the $\mathscr{P}$-positions are circled. Plus and minus signs indicate the options; a query, ?, means that the option wins more slowly or loses more quickly than necessary; and ?? means that the option loses when the alternative would give a win.

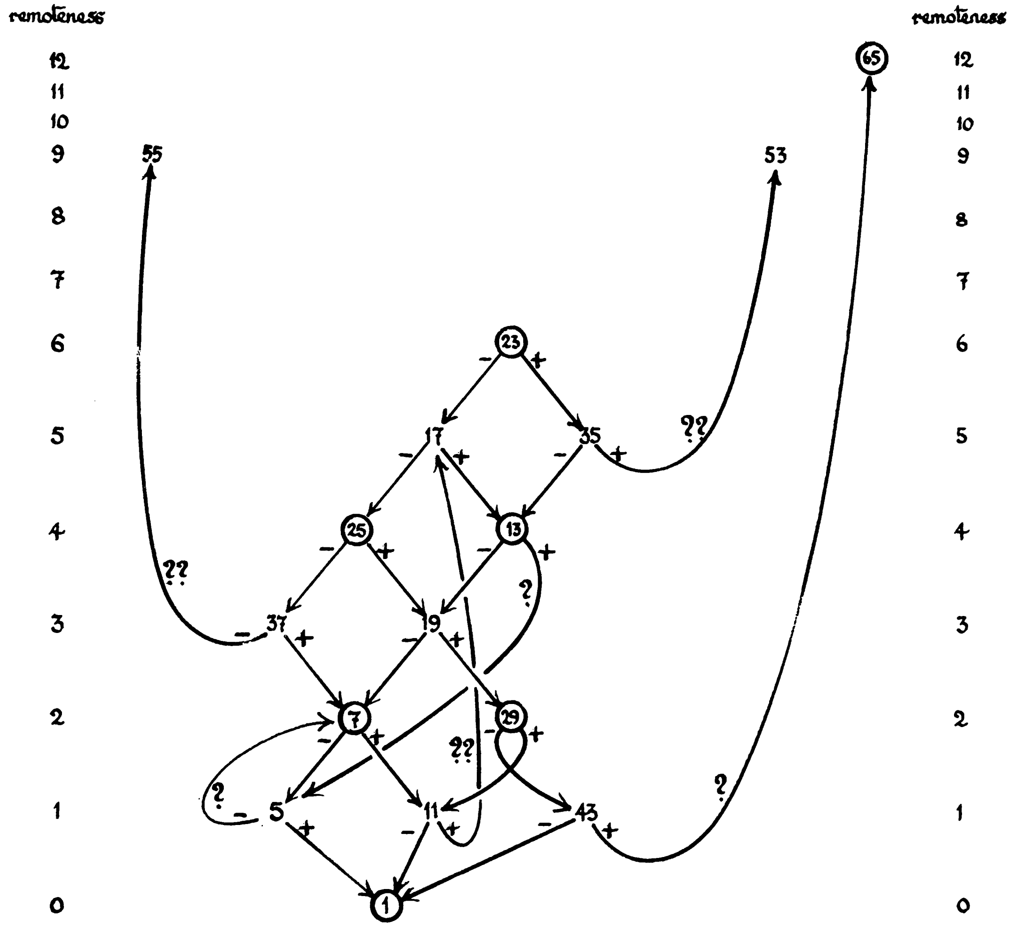


FIGURE 2. **Plays of Beans-Don't-Talk, starting from the 𝒫-position 23.**

The remoteness of a position is calculated recursively from the remotenesses of the options. The remoteness of a terminal position (e.g., 1 in Beans-Don't-Talk) is 0. If there's an option with *even* remoteness, then you're in an 𝒩-position, and the remoteness is one more than the *least even* remoteness of an option. If all the options have *odd* remoteness, you're in a 𝒫-position, and the remoteness is one more than the *greatest* (odd) remoteness of an option.

Calculating Remotenesses

1 more than least even (win quickly!) ELSE 1 more than greatest odd (lose slowly!) ELSE 0 (terminal position).

In Beans-Don't-Talk, the positions of remoteness 1 are $\{2^{a+2}-(-1)^a\}/3$, for $a \geqslant 1$, i.e., Sloane [4, seq. 983]:

$$3,5,11,21,43,85,171,341,683,1365,2731,5461,\ldots.$$

Those of remoteness 2 are $\{2^{3a+3}+(-1)^a\}/9$ and $\{2^{3a+5}-5(-1)^a\}/9$, for $a \geqslant 1$, i.e.,

$$7, 57, 455, 3641,\ldots \quad \text{and} \quad 29, 227, 1821, 14563,\ldots.$$

Conway noticed that the positions of given remoteness fall into **chunks**, the numbers in any chunk after the first being roughly twice those in the previous chunk. For example, the positions of remoteness 6 are

$$\begin{array}{c} 6 \\ 12 \\ 23 \\ 45,46 \\ 89,91,92,93 \\ 177,179,183,185 \\ 354,355,359,367 \\ 707,708,709,711,717,718,719,733,739 \\ 1415,1417,1433,1435,1437,1438,1465,1469,1479 \\ \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

It's easy to see why this is so. Can we use it to speed up the analysis? We saw a similar phenomenon in Beanstalk; in TABLE 1 there are chunks of equal lengths. This is because the lengths are indeed remotenesses, but all odd, $\mathcal{N}$-positions or Jack-winning positions.

TABLE 2 lists some positions of remoteness 3, 4 or 5.

Remoteness 3:	2,9,10,19,37,39,75,76,77,149,151,152,155,299,303,309,597,605,607,619,1195,1211, 1213,1214,1237,2389,2421,2427,2475,4779,4843,4853,4854,4855,4949,9557,...
Remoteness 4:	13,25,50,51,99,101,103,199,202,403,404,405,413,797,807,809,825,1593,1618,3229, 3235,3236,3237,3299,6371,6457,6471,6473,6599,...
Remoteness 5:	4,8,17,33,34,35,66,67,69,133,134,135,137,138,139,265,266,267,269,270,275,277, 531,533,537,539,549,551,555,1061,1063,1067,1075,1076,1077,1078,1079,1099,1100, 1101,1109,2123,2124,2125,2133,2149,2152,2153,2155,2157,2158,2197,2199,2200,2203, 2219,4245,...

TABLE 2. **Positions of remoteness 3, 4, or 5.**

Conway also noticed that certain numbers have derivates. The options from $8n+1$ are $12n+1$ and $6n+1$. From each of these we can move to $(9n+1)/2^*$, which is the **derivate** of $8n+1$. Similarly, from either option of $8n-1$, we can move to *its* derivate $(9n-1)/2^*$. In the same way, $16n \pm 3$ have respective derivates $(9n \pm 2)/2^*$. Derivates are useful in the analysis, since a number whose derivate is a $\mathcal{P}$-position is itself a $\mathcal{P}$-position.

TABLE 3 (also hand-calculated, so that checking and extension are again desirable) shows the remoteness of the first 500 numbers in Beans-Don't-Talk. The $\mathscr{P}$-positions and $\mathscr{N}$-positions occur in longer and longer consecutive runs, whose remotenesses are all in the same residue class, mod 12. The reason for 12 is, once again, the good approximation $3^{12} \approx 2^{19}$. The residue class of each run is 5 more than that of the preceding run: the reason is the good approximation $3^5 \approx 2^8$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	0	3	1	5	1	6	2	5	3	3	1	6	4	9	7	24	5	10	3	8	1	13	6	11	4
25 +	9	9	26	2	7	7	24	5	5	5	22	3	15	3	8	8	25	1	30	6	6	23	11	11	4
50 +	4	33	9	9	9	26	2	31	7	7	7	7	24	12	12	5	5	17	5	34	22	10	10	27	3
75 +	3	3	32	8	20	8	8	25	25	1	13	13	18	6	18	6	6	6	11	23	11	11	11	4	28
100 +	4	16	4	33	9	9	21	9	9	9	26	26	26	14	14	31	31	7	7	7	7	7	7	36	24
125 +	24	24	24	12	12	29	17	5	5	5	17	5	5	5	10	10	34	22	22	10	10	27	27	3	27
150 +	3	3	15	15	3	32	32	8	8	20	20	20	8	8	8	13	25	13	25	13	1	13	13	13	30
175 +	30	6	30	6	18	18	18	6	35	6	11	11	11	11	23	11	11	11	11	11	40	28	28	4	28
200 +	28	4	16	16	16	16	33	21	33	9	9	9	9	21	9	9	9	9	9	38	26	14	26	14	26
225 +	26	2	26	14	14	14	43	31	19	31	7	7	31	7	7	19	19	7	7	7	7	7	36	12	12
250 +	24	12	24	24	12	24	24	12	12	12	12	17	17	29	5	5	5	29	5	5	17	17	17	17	5
275 +	34	5	34	34	10	10	34	10	34	34	22	10	22	22	10	10	10	10	15	27	15	27	15	3	27
300 +	27	15	3	27	15	15	15	15	3	44	32	20	32	32	8	8	8	32	8	32	20	20	8	20	20
325 +	8	8	37	8	37	13	13	13	13	13	13	25	25	25	13	1	25	13	13	13	13	13	42	30	30
350 +	18	30	30	6	6	30	30	30	6	30	18	18	18	18	18	18	6	35	35	35	35	23	11	11	11
375 +	11	11	11	23	23	23	23	11	23	23	11	11	11	11	11	11	40	28	28	16	28	28	16	28	28
400 +	28	28	4	4	4	28	16	16	16	16	16	16	4	33	21	21	33	33	9	9	9	9	9	9	9
425 +	9	33	21	21	21	9	21	21	9	9	9	9	9	9	38	38	14	26	14	26	38	14	14	26	26
450 +	26	26	26	14	2	26	26	14	14	14	14	14	14	43	31	31	19	19	31	31	7	7	7	31	7
475 +	31	7	7	7	31	19	19	19	19	7	19	19	7	7	7	7	36	7	48	36	36	12	12	12	12

TABLE 3. **Tentative table of remotenesses for Beans-Don't-Talk.**

What is the probability, τ, that a number is an $\mathscr{N}$-position? Assume that there are no $\mathscr{O}$-positions, or at least that their density is zero. Then the probability that a number is a $\mathscr{P}$-position is $1 - \tau$. This happens just if *both* options are $\mathscr{N}$-positions, so that $1 - \tau = \tau^2$ and τ is the **golden ratio** $(\sqrt{5} - 1)/2 \approx 0.618$. Jeff Lagarias asked if this was supposed to be a proof, or merely a heuristic argument. Let's look at the evidence. Table 3 shows that 5 of the first 8, 8 of the first 13, 13 of the first 21, 21 of the first 34, and 34 of the first 55 numbers are $\mathscr{N}$-positions. Pretty convincing? But recall Conway's observation about derivates: the two options are far from being independent: their status is more likely to be the same as not. Our argument is not justified: the proportions of $\mathscr{N}$-positions is not likely to be as high as we have suggested. The Fibonacci phenomenon that we've just observed is another example of the Strong Law of Small Numbers [**2**]. Only 52 of the first 89 numbers, 81 of the first 144, 126 of the first 233, and 201 of the first 377 are $\mathscr{N}$-positions (deficiences 3, 8, 18, and 32 from our Fibonacci rule). Indeed, the lengths of the chunks, even the sizes of the deficiences, appear to be increasing exponentially. It may even be more reasonable to conjecture that the probability, in the sense of an exact limit, does not exist!

The same argument, with similar reservations, can be given for any game in which there are just two options. Its validity varies considerably, as we can see from three more examples.

Richard Epstein's game of Put-or-Take-a-Square [**1**, pp. 484–486, 501–502] is played with one heap of beans. The move is to take from or add to the heap the largest square number of beans in the heap. So squares are $\mathscr{N}$-positions of remoteness 1, since the object is to be the player who takes the last bean. Examples of $\mathscr{P}$-positions are 5, 20, and 29. The great majority of positions $(2,3,6,7,8,10,12,13,15,17,\dots)$ are $\mathscr{O}$-positions: this clouds the issue of the ratio of numbers of $\mathscr{N}$-positions to $\mathscr{P}$-positions. For the first few hundred numbers, this seems to be close to 3.

Simon Norton's game of Tribulations [**1**, pp. 486, 501] is like Epstein's game, but instead of the largest square you subtract or add the largest triangular number $(1,3,6,10,15,21,\dots)$. Norton conjectures that there are no $\mathscr{O}$-positions, and that the $\mathscr{N}$-positions are more numerous than $\mathscr{P}$-positions in golden ratio. Richard Parker has verified this for numbers less than 5000.

In Mike Guy's game of Fibulations the subtrahend or addend is the largest among the Fibonacci numbers plus one $(1,2,3,4,6,9,14,22,\dots)$. Here the analysis is complete [**1**, pp. 486, 501–503] and the corresponding assertions can be proved.

Would a solution to Beanstalk or Beans-Don't-Talk throw any light on the Collatz problem: does iteration of the function $(3x+1)/2^*$ always lead to 1? The existence of drawn positions in either game would imply cycles or infinite sequences of the kind that are conjectured not to exist for the $3x+1$ problem.

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When Ralph Boas lectured at a seminar in Cambridge, he did not try to change his verbiage to the British usage of "epsilon dash, alpha n, zed." He concentrated on the mathematics. It was one of his first teaching assignments before an advanced audience. So one of his listeners wrote the following limerick on the blackboard:

It's really quite easy to see
That a man's from the "land of the free"
When he talks all the time
About epsilon prime,
Alpha sub-*n*, zee, and phee.

Boas told me this story himself.

—Joel Brenner
Palo Alto, California

NOTES

Pairs of Squares with Consecutive Digits

C. B. LACAMPAGNE
University of Michigan-Flint
Flint, MI 48503

J. L. SELFRIDGE
Northern Illinois University
DeKalb, IL 60115

In a problem-solving lesson, Mary Grace Kantowski posed the following problem to sixth grade students: Find all two- and three-digit squares such that when one is subtracted from each digit, the resulting number is a square. The students quickly found the solution for the two-digit problem through trial and error: the square is 36. Checking for a three-digit solution by hand seemed onerous, but by writing and executing a simple computer program, the youngsters found that no three-digit solution exists.

The natural extension of n-digit squares is the subject of this note.

THE PROBLEM. *For each n, find all positive integers x and y such that $x^2 - y^2 = 11 \cdots 1$ (n ones), and x^2 contains no zero digits.*

In what follows, we list numbers of solutions in bases 10, 3 and 4, for $n \leq 125$, 18, and 83, respectively. Further, we propose a probability model that suggests that no solutions exist for n sufficiently large. Lastly, we mention the possibility of finding some "magic" base for which infinitely many solutions exist. We thank Richard Guy for his help, especially with bases 3 and 4, and Paul Erdős for suggesting that our theorem is an interesting result which surprisingly does not seem to be in the literature.

The attack

We began by doing a computer search for solutions in base 10 (see TABLES 1 and 2). From scanning the data that we generated, it appeared that solutions become rare as n becomes large. We wondered whether for n greater than a certain number N, no solutions exist. Yet predicting the largest solution seemed elusive, as TABLE 2 indicates.

Since the problem is obviously base-dependent, we decided to look at bases other than 10. With fewer distinct digits available, perhaps the largest solution could be computed. Certainly base 2 is trivial, since any square with more than one digit has zeros. So we looked at base 3 (TABLE 3). As expected, computation in base 3 yielded few solutions. Only five solutions were found, the largest being

$$(202_3)^2 - (20_3)^2 = 111111_3.$$

Perhaps some reader can prove that the number of solutions is finite for base 3.

Next we looked at base 4 (TABLE 4). After finding a solution for $n = 18$ and none for $n = 19$ through 39, we were quite surprised that a solution appeared for $n = 40$. Looking at our probability model, we found that this was totally unexpected. We continued our computer run through $n = 83$, finding no further solutions. However, the reader who enjoys programming may wish to carry our calculations further or to work with other bases.

n	b	a	x	y	x^2	
1	1	1	1	0	1	
2	1	11	6	5	36	
3	3	37	20 not counted since x ends in zero			
4	11	101	56	45	3136	
5	41	271	156	115	24336	
6	231	481	356	125	126736	plus 2 bad
	259	429	344	85	118336	solutions
7	239	4649	2444	2205	5973136	
8	1111	10001 is the one bad solution (x^2 = 308...)				
9	nothing since 333667 is prime					
10	two bad solutions; x^2 = 43107... and 308...					
11	21649	513239	267444	245795	71526293136	
12	108911	1020201	564556	455645	318723477136	plus 3
	207921	534391	371156	163235	137756776336	bad
	287749	386139	336944	49195	113531259136	solutions
13	nothing; 265371653 is a prime factor > B_{10}					
14	two bad solutions; x^2 = 43107... and 308...					
15	2906161	38232951 is the one bad solution (x^2 = 423106...)				
16	11111111	100000001 is the one bad solution (x^2 = 308...)				
17	nothing; 5363222357 is a prime factor > B_{10}					
18	122561649	906573239	514567444	plus		
	230769231	481481481	356125356	14		
	258741259	429429429	344085344	bad		
	259259259	428571429	343915344	solutions		
19	1111111111111111111 is prime					
20	765320531	14518245181	7641782856	plus		
	1089108911	10202020201	5645564556	5		
	1273651511	8723823601	4998737556	bad		
	2737127371	4059405941	3398266656	solutions		
21	2590446671	42892645641	22741546156	plus 2 bad solutions		
22	22418167321	49562976991	35990572156	plus 3 bad solutions		
23	11111111111111111111111 is prime					
24	210021210021	529047095291	369534152656	plus 20 bad		
	245791245791	452054794521	348923020156	solutions		

TABLE 1. **Base-10 solutions for small *n*.**

n	good	total	n	good	total	n	good	total
25	0	1	56	0	33	90	≥ 1	32386
26	0	3	58	0	2	91	0	38
28	0	15	60	≥ 11	19425	92	0	9
30	23	237	62	0	2	94	0	2
32	4	54	63	0	151	96	≥ 1	42965
34	0	2	64	1	409	98	0	5
35	0	1	66	1	269	100	0	2538
36	2	41	68	0	13	102	0	543
38	0	2	70	0	97	104	0	33
40	1	41	72	3	2336	105	0	988
42	11	547	74	0	4	106	0	2
44	0	30	75	0	4	108	0	3522
45	0	7	76	0	6	110	0	1598
46	0	2	78	0	375	112	0	757
48	1	94	80	1	217	114	0	40
50	0	30	81	0	14	116	0	40
51	0	4	82	0	2	118	0	2
52	0	8	84	≥ 1	31942	120	?	331525
53	0	1	85	0	1	122	0	3
54	1	112	86	0	4	124	0	3
55	0	2	88	0	34			

TABLE 2. **Number of base-10 solutions $25 \leq n \leq 125$.**

To write an efficient computer program for finding solutions, we needed to look for constraints on our variables. Since

$$x^2 - y^2 = (x+y)(x-y) = 11\cdots 1 \quad (n \text{ ones}),$$

the problem is equivalent to factoring n ones into all possible pairs of factors a and b, such that $x+y=a$, $x-y=b$, and, hence, $x=(a+b)/2$.

But considering as small a case as $n=6$ in base 10 where

$$x^2 - y^2 = 111111 = 3\cdot 7\cdot 11\cdot 13\cdot 37,$$

there are 16 pairs of complementary subsets of $\{3,7,11,13,37\}$ and, therefore, 16 ways in which we can combine the five prime factors of 111111 to get potential pairs of factors with $b<a$. By finding bounds for a and b we eliminate some of the potential factor pairs. Notice that $x^2=((a+b)/2)^2<10^6$, $ab=111111$, and by substituting and solving the resulting quadratic inequality, we find that $a<1943$, eliminating 9 of the possible 16 pairs of factors.

In general, to obtain bounds on the factors a and b, let r be the base. Then

$$b \le a, \qquad x^2 = ((a+b)/2)^2 < r^n$$

and

$$ab = (r^n - 1)/(r-1).$$

Eliminating b and solving the resulting quadratic inequality for a, we find an upper bound

$$B_r < \left(1+\sqrt{(r-2)/(r-1)}\,\right) r^{n/2}.$$

Thus for bases 10, 3, and 4, respectively, $B_{10}=1.9428\cdot 10^{n/2}$, $B_3=1.7071\cdot 3^{n/2}$, and $B_4=1.8165\cdot 4^{n/2}$.

Note that not all x's thus generated are "good" solutions, for often x^2 contains zeros as digits. We will call those x's with zeros in x^2 "bad" solutions. In our example, when $a=3\cdot 13\cdot 37$ and $b=7\cdot 11$, then $a+b=1443+77=1520$. So $x=1520/2=760$ and x^2 ends in zeros. In fact, any pair of factors a and b in our example can be excluded from consideration if $a+b\equiv 0 \pmod{20}$. Three more possible pairs are eliminated in this manner, leaving only four pairs to check, two of which have an internal zero digit. Two genuine solutions remain where $b=3\cdot 7\cdot 11$, $a=13\cdot 37$; and $b=7\cdot 37$, $a=3\cdot 11\cdot 13$ (see TABLE 1).

In general, if $a+b\equiv 0 \pmod{2r}$, x^2 will end in a zero. Such solutions may be eliminated from consideration before x^2 is calculated and will not appear in our count of bad solutions. In base 10 let $a\equiv 3,7,13,17 \pmod{20}$. Then since $ab\equiv 11 \pmod{20}$, $b\equiv 17,13,7,3 \pmod{20}$, respectively. In all cases, $a+b\equiv 0 \pmod{20}$. Thus in base 10, the conditions $ab\equiv 11$ and $a+b\not\equiv 0 \pmod{20}$ exclude all factors a and b ending in 3 or 7.

A program was written and executed to generate solutions in each of the three bases. The program included a backtrack subroutine to find all pairs of factors a and b with $a<B_r$ which would not have a zero units digit in x^2, and in the case of base 10 would have correct units digits for a and b. The program included subroutines for multiple-precision product, addition and halving, and dealt with numbers having hundreds of digits. Prime factors of $(r^n-1)/(r-1)$ were entered manually from the tables of Brillhart, Lehmer, Selfridge, Tuckerman, and Wagstaff [**1**]. TABLES 1–4 are based on our output.

The largest good solution found for n odd was $x^2=517177921565478376336$ ($n=21$). The largest solution found on our TRS-80 (model 1) had 96 digits:

$$x^2 = 134128\,446923\,678934\,432974\,269833\,336334\,559656\,998355$$
$$535334\,715749\,995466\,956572\,828193\,474637\,363129\,723136.$$

(In TABLE 2, n is omitted when there are no solutions, good or bad.)

According to the probability model (which we describe later), there is about a 77% chance that one (or more) of the 331525 solutions for $n=120$ is a good solution. Our computer is too slow to check these in a reasonable amount of time.

n	b	a	x	y	x^2
1	1	1	1	0	1
2	2	2	2	0	11
3	$(111)_3 = 13$ is prime				
4	11	101	21	10	1211
5	102	102	102	0	11111
6	112	222	202	20	112211
7	$(1111111)_3 = 1093$ is prime				
8	1111	10001	2021 is the one bad solution		
9	$(1001001)_3 = 757$ is prime				
10	11112	22222	20202 is the one bad solution		
11	3851 is a prime factor > B_3				
12	three bad solutions				
13	$(1111111111111)_3 = 797161$ is prime				
14	1111112	2222222	2020202 is the one bad solution		
15	2011021	20020221	11012121 is the one bad solution		
16	five bad solutions				
17	34511 is a prime factor > B_3				
18	three bad solutions				

TABLE 3. **Base-3 solutions.**

n	b	a	x	y	x^2	
1	1	1	1	0	1	
2	1	11	3	2	21	
3	3	13	11	2	121	
4	11	101	23	12	1321	
5	23	133	111	22	12321	
6	31	1221	323	232	312121	plus
	33	1123	311	212	223321	1 bad
	203	213	211	2	111121	solution
7	223	1333 is the one bad solution				
8	1111	10001 is the one bad solution				
9	three bad solutions					
10	3031	122221	32323	23232	313222121	plus 2
	13003	30013	21211	2202	1123233121	bad solutions
11	22223	133333 is the one bad solution				
12	120023	320133	220111	100022	121122112321 + 5 bad	
13	222223	1333333 is the one bad solution				
14	303031	12222221	3232323	2323232	3132332322121 + 4 bad	
15	six bad solutions					
16	11111111	100000001 is the one bad solution				
17	2222223	13333333 is the one bad solution				
18	210110301	211211211	210331023	220122	1111112323113333321 + 38 bad	
19 to 39	634 bad solutions					
40	b = 20311330332031133033		a = 21221211212212112123			
	x = 21033301102121322311		y = 1213101100301232l2			
	x^2 = 1111133233313313331131222233213132233321 + 9 bad solutions					
41 to 83	10923 bad solutions					

TABLE 4. **Base-4 solutions.**

The number of factors of $(r^n - 1)/(r-1)$

It is clear that the total number of solutions $T(n)$ of $ab = N = 11 \cdots 1$ (n ones) depends on the number of prime factors of N. If there are k such factors (counting multiplicity), then clearly $T(n) \le 2^{k-1}$, the number of pairs of complementary subsets of k elements. The theorem below asserts that for n large, the number of these prime factors k is very small compared to n. This result is used in the next section to estimate the number of good solutions.

THEOREM. *Let k be the number of prime factors (counting multiplicity) of $r^n - 1$. Then*

$$\lim_{n\to\infty} k/n = 0.$$

Proof. We estimate first the number t of small prime factors and then the number u of large prime factors. Consider a prime p to be small if $p \le \sqrt{n}$. Suppose $p | r^n - 1$. Let h be the least positive exponent such that $r^h \equiv 1 \pmod p$, and let x be the largest exponent such that p^x divides $r^h - 1$. (It is usual to write $p^x \| r^h - 1$ and to say "exactly divides.") Since h is the least positive exponent such that $p | r^h - 1$ and $p | r^n - 1$, then $h | n$. Define s by $hp^s \| n$. Since $p^x \| r^h - 1$, it follows easily from the binomial theorem that $p^{x+1} \| (r^h)^p - 1$ and by repeating the argument, we get $p^{x+i} \| (r^h)^{p^i} - 1$. From this it is clear that the order of $r \pmod{p^{x+i}}$ is $h \cdot p^i$. This shows that $r^n \equiv 1 \pmod{p^{x+s}}$ and that since $h \cdot p^{s+1} \nmid n$, $r^n \not\equiv 1 \pmod{p^{x+s+1}}$; this gives $p^{x+s} \| r^n - 1$.

Now we estimate. Since $h | p-1$, $p^x < r^{p-1}$ and taking logarithms, $x < p \ln r / \ln p$. Since $hp^s \le n$, $s \le \ln n / \ln p$. Hence the number t of small prime factors of N is at most

$$\begin{aligned}\Sigma(x+s) &= \Sigma x + \Sigma s \\ &< \ln r \Sigma p/\ln p + \ln n \Sigma 1/\ln p\end{aligned}$$

where each sum is taken over all primes less than $\sqrt{n}$. Now according to the prime number theorem, there are approximately $\sqrt{n}/\ln\sqrt{n} = 2\sqrt{n}/\ln n$ primes up to $\sqrt{n}$, more precisely, $(1+\varepsilon(n))\sqrt{n}/\ln\sqrt{n}$, where the error term $\varepsilon(n)$ goes to zero as n becomes large.

Multiplying the number of terms by the largest term in the summations above, we get

$$\Sigma(x+s) < (\ln r)(1+\varepsilon(n))\left(\sqrt{n}/\ln\sqrt{n}\right)^2 + c_1\sqrt{n} < c_2 n/(\ln n)^2,$$

where c_2 can be taken as $(4+\varepsilon_1(n))\ln r$ for n large enough.

Now consider the large prime factors of N. Let $q_1 \le \cdots \le q_u$ be all prime factors of N greater than $\sqrt{n}$. Then

$$(\sqrt{n})^u < q_1^u < \prod_{i=1}^{u} q_i \le (r^n - 1)/(r-1) < r^n.$$

Taking logarithms, we get $u < 2n \ln r/\ln n$. Finally,

$$k/n = t/n + u/n < c_2/(\ln n)^2 + 2\ln r/\ln n,$$

thus proving that k/n goes to zero as n becomes large.

Estimation of the number of good solutions

In scanning the TABLES, it seems that the number of good solutions becomes scarce for large n, and the question arises, "Can we estimate the number of good solutions?" Let $E(n)$ be the expected number of good solutions to the n-digit problem. Good solutions have nonzero digits in x^2. We assume that the probability of an internal digit of x^2 being nonzero is $(r-1)/r$. If we further assume that these probabilities are independent, we get

$$E(n) = c((r-1)/r)^n T(n) \le c((r-1)/r)^n 2^{k-1}.$$

But by the Theorem, $\lim_{n\to\infty} c((r-1)/r)^n 2^k = 0$. Thus for n large we would not expect to find a good solution. Moreover, by the Theorem, for n large enough, $2^{k/n} < r/(r-1)$, and, so, $\Sigma E(n)$ can be compared to a convergent geometric series $c\Sigma y^n$ with $y<1$. Thus there are probably only a finite number of good solutions for a given base r. The independence assumption is clearly unwarranted for the final three digits (base 10), but we see no reason to doubt it in general.

For example, there are 55 factors of 420 1's (base 10). Since $(9/10)^{20} < 1/8$, we have $(9/10)^{420} < (1/2)^{63}$, and we get $E(420) < c(1/2)^{63}2^{54} < 0.002c$. Thus it would be unlikely to find a good solution to the 420-digit problem.

On the other hand, there are only nine factors of 40 1's (base 4) and a good solution exists despite the low probability $E(40) < c(3/4)^{40}2^8 < 0.0026c$.

Despite the improbability of finding solutions for large n in any given base r, there might be a pattern in some "magic" base which generates an infinite number of solutions. The base would have a factorization pattern for certain values of n 1's such that the resulting x^2 would contain no zero digits. We certainly have no way of proving that such a base does not exist. On the contrary, we decided to search for such a base. Our quest was unsuccessful. Perhaps some reader will settle the question.

Reference

[1] John Brillhart, D. H. Lehmer, J. L. Selfridge, Bryant Tuckerman, and S. S. Wagstaff, Jr., Factorizations of $b^n \pm 1$, Contemp. Math., 22, Amer. Math. Soc., 1983.

The Arithmetic of Differentiation

L. B. RALL
Mathematics Research Center
University of Wisconsin-Madison
Madison, WI 53706

Traditional ways of thinking and doing things exert a strong hold in mathematics, as well as other human endeavors. For example, generations of students were taught to solve linear systems of equations by using determinants and Cramer's rule [**2**]. It was not until people actually tried solving linear systems using desk calculators and, later, computers that more efficient methods based on Gaussian elimination came into general use [**3**]. Even today, it is disconcerting to encounter people who think that the "only way" to solve linear systems is by the Cramer's rule they learned in school.

Calculus is another subject with a long, rich history and strong traditions. As an example of a traditional approach to a simple problem in calculus, suppose you are given a function defined by a formula, say

$$f(x) = \frac{x^2+2x-3}{x+2} = \frac{(x-1)\cdot(x+3)}{x+2}, \tag{1}$$

and you want to find the value of the derivative $f'(x)$ for a given value of x. Most people have been taught to solve this problem by first deriving a formula for the derivative,

$$f'(x) = \frac{x^2+4x+7}{(x+2)^2} = 1 + \frac{3}{(x+2)^2}, \tag{2}$$

and then using the usual rules of arithmetic to evaluate $f'(x)$. This method, like Cramer's rule, is good solution. Moreover, by the Theorem, for n large enough, $2^{k/n} < r/(r-1)$, and, so, $\Sigma E(n)$ can be compared to a convergent geometric series $c\Sigma y^n$ with $y < 1$. Thus there are probably only a finite number of good solutions for a given base r. The independence assumption is clearly unwarranted for the final three digits (base 10), but we see no reason to doubt it in general.

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$$f'(x) = \frac{x^2 + 4x + 7}{(x+2)^2} = 1 + \frac{3}{(x+2)^2}, \tag{2}$$

and then using the usual rules of arithmetic to evaluate $f'(x)$. This method, like Cramer's rule, is theoretically correct, and not complicated for this example. However, this is not the only way to calculate the values of derivatives. It turns out that it is possible to obtain both the values $f(x)$ and $f'(x)$ using a different kind of arithmetic called differentiation arithmetic. A formula for $f'(x)$ is not required, contrary to common belief. Differentiation arithmetic can be formulated as an ordered-pair arithmetic, similar to complex arithmetic. The rules for this arithmetic will be defined later; first, it is important to analyze the key idea of function evaluation in ordinary arithmetics, and see what this process has in common with differentiation.

The usual way to evaluate a rational function, (1), for example, is to break it down into a sequence of additions, subtractions, multiplications, and divisions, and then apply the rules of arithmetic to each. Thus, the only difference in the evaluation of $f(3)$ and $f(2-3i)$ is that the rules of real arithmetic are used for the first, while the second is obtained by application of the rules of complex arithmetic.

Differentiation of functions is of course based on the definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{3}$$

However, this definition is seldom used for actual differentiation of a function such as $f(x)$ given by (1). Instead, it is used to derive *rules* for differentiation of sums, differences, products, quotients, and other functions encountered in mathematical analysis. Once these rules have been established, differentiation proceeds in much the same way as function evaluation. The function is expressed as a sequence of arithmetic operations and other functions with known derivatives, and the appropriate rules are applied. This technique is valid as a consequence of the *chain rule*, which is derived from the definition (3). This is the way in which the *formula* (2) for $f'(x)$ is derived from the formula (1) for $f(x)$.

Now, in order to find $f'(3)$, for example, the usual way to proceed is first to derive the formula (2), and then evaluate it at $x=3$ by breaking it down into a sequence of arithmetic operations and applying the rules of real arithmetic. What will be shown here is a method which obtains the correct values of $f(3)$ and $f'(3)$, for example, directly from the formula (1) which defines $f(x)$. This process is called **automatic differentiation** [7] to distinguish it from *symbolic differentiation*, which produces *formulas* such as (2), and *numerical differentiation*, which yields only approximate values for $f'(x)$, for example, by use of the difference quotient

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{4}$$

for a fixed value of Δx. Automatic differentiation eliminates the steps of production and evaluation of the formula for $f'(x)$ in symbolic differentiation, without introducing the truncation error inherent in numerical differentiation. In fact, (3) defines the *value* of $f'(x)$, and not a formula for the derivative of the function f, so there is nothing inherently symbolic in the differentiation process.

Differentiation arithmetic

The idea behind differentiation arithmetic is the familiar one of an *ordered-pair* arithmetic. Here, arithmetic operations are defined in the set $\mathbf{S}^2 = \mathbf{S} \times \mathbf{S}$ of pairs of elements of a set $\mathbf{S}$ componentwise in terms of already established operations in $\mathbf{S}$. Some examples, with typical elements having components in $\mathbf{S} = \mathbf{Z}$, the set of integers, or $\mathbf{S} = \mathbf{R}$, the set of real numbers, are:

(i) *rational* arithmetic: $r = p/q, \quad p, q \in \mathbf{Z}$;

(ii) *complex* arithmetic: $z = (x, y) = x + iy, \quad x, y \in \mathbf{R}$;

(iii) *interval* arithmetic: $I = [a, b] = \{x \mid a \leqslant x \leqslant b;\ a, b, x \in \mathbf{R}, \quad a \leqslant b\}$.

The rules of rational and complex arithmetic are well-known; interval arithmetic is based on the concept

$$[a, b] * [c, d] = \{x * y \mid x \in [a, b], y \in [c, d]\},$$

where $*$ denotes one of the arithmetic operations $+, -, \times, /$ (see [1] or [6] for more details about interval arithmetic and its applications). In any of these arithmetics, $f(x)$ can be evaluated by applying the rules appropriate for the type of x to the same sequence of arithmetic operations which give $f(x)$ for x real.

The basic **elements** for **differentiation arithmetic** are pairs of real numbers,

$$U=(u,u'), \qquad V=(v,v'),\ldots \in \mathbf{R}^2.$$

For the time being, the prime is to be regarded only as a marker to distinguish the second element of a pair. The **rules** of differentiation arithmetic are as follows:

1°. Addition

$$U+V=(u,u')+(v,v')=(u+v,u'+v'); \tag{5}$$

2°. Subtraction

$$U-V=(u,u')-(v,v')=(u-v,u'-v'); \tag{6}$$

3°. Multiplication

$$U\cdot V=(u,u')\cdot(v,v')=(u\cdot v,u\cdot v'+v\cdot u'); \tag{7}$$

4°. Division

$$\frac{U}{V}=\frac{(u,u')}{(v,v')}=\left(\frac{u}{v},\frac{v\cdot u'-u\cdot v'}{v^2}\right), \qquad v\neq 0. \tag{8}$$

Examination of the rules shows their simple structure. The first components of the results are formed by the rule for *evaluation* in real arithmetic, and the second by the corresponding rule for *differentiation*, if we assume that the first components represent function values, and the second components the values of their derivatives. The set $\mathbf{R}^2$ with the operations (5)–(8) will be denoted by $\mathbf{D}$. The real number system $\mathbf{R}$ with ordinary arithmetic is embedded isomorphically into $\mathbf{D}$ by the mapping $c \mapsto (c,0)$, $c\in\mathbf{R}$.

A rational function $f:\mathbf{R}\to\mathbf{R}$ can be extended to a function $f:\mathbf{D}\to\mathbf{D}$ by the use of differentiation arithmetic in exactly the same way the extension of f to a complex function $f:\mathbf{C}\to\mathbf{C}$ is carried out by complex arithmetic. More precisely, let $f((u,v))$ denote the ordered pair $(s,t)\in\mathbf{D}$ obtained by substituting $(u,v)\in\mathbf{D}$ for x and $(c,0)\in\mathbf{D}$ for each constant c in the algebraic expression for f and using the rules (5)–(8) for differentiation arithmetic.

Note that the same symbol f and a formula such as (1) for a rational function can represent a number of distinct functions, depending on the type of the argument x. The type of x generally determines the kind of arithmetic to be used and the type of the result obtained. Usually, it is possible to determine from context exactly how f is to be interpreted. The connection between the arithmetic in $\mathbf{D}$ defined formally by (5)–(8) and the evaluation of derivatives becomes clear if $X=(x,1)$ is substituted for x and $C=(c,0)$ for constants c in the expression for a rational function f. These values correspond to the rules of calculus

$$\frac{dx}{dx}=1, \qquad \frac{dc}{dx}=0,$$

for differentiation of the independent variable and constants, respectively. It follows from the chain rule that substitution of X, C into a formula for a rational function will give

$$f(X)\equiv f((x,1))=(f(x),f'(x)),$$

and thus yields both the value of the *real* function f and its derivative f' at $x\in\mathbf{R}$.

For example, substitution of $X=(3,1)$ and appropriate values for the constants in (1) gives:

$$f((3,1))=\frac{((3,1)-(1,0))\cdot((3,1)+(3,0))}{(3,1)+(2,0)}=\frac{(12,8)}{(5,1)}=\left(\frac{12}{5},\frac{28}{25}\right).$$

That is,

$$f(3)=\frac{12}{5}, \qquad f'(3)=\frac{28}{25}.$$

The important thing is that these values were obtained directly by using (1) and differentiation arithmetic; formula (2) was never produced.

Differentiation arithmetic is easy to program for computers [**7**], and even hand-held programmable calculators. In the case of computers, it is particularly convenient to program differentiation arithmetic in languages such as ALGOL 68, Ada, or Pascal-SC [**8**] which permit "overloading" the standard operator symbols. This means, for example, that the meaning of "+" can be defined independently for various types of operands (integer, real, complex, interval, etc.) in such languages. In actual computation, it is cumbersome to carry constants in the form $(c,0)$. To simplify matters, a *mixed* arithmetic can be defined between real numbers and elements of **D**, with rules similar to the rules for arithmetic between real and complex numbers [**8**].

The properties of the arithmetic in **D** defined by (5)–(8) can be examined independently of its connection with differentiation. It is easy to verify the following:

Addition in D forms a commutative group. The identity element for addition is

$$\mathbf{0}=(0,0),$$

and each element $U\in\mathbf{D}$ has the additive inverse (or negative)

$$-U=-(u,u')=(-u,-u').$$

Multiplication in D forms a commutative semigroup with an identity, that is, multiplication is commutative and associative, and

$$\mathbf{1}=(1,0)$$

is the identity element for multiplication, $U\cdot\mathbf{1}=\mathbf{1}\cdot U=U$ for all $U\in\mathbf{D}$.

Multiplication is distributive across addition,

$$U\cdot(V+W)=U\cdot V+U\cdot W,$$

for all $U,V,W\in\mathbf{D}$.

It follows that **D is a commutative ring with an identity element** [**4**]. Division is defined in **D** except for elements of the form $(0,r)$, where r is an arbitrary real number; these elements of **D** are called nonunits. If $u\neq 0$, then

$$(u,u')\cdot(v,v')=\mathbf{0}\Leftrightarrow(v,v')=(0,0)=\mathbf{0}.$$

However, for arbitrary nonunits $(0,r)$, $(0,s)$,

$$(0,r)\cdot(0,s)=(0,0)=\mathbf{0}, \tag{9}$$

so that **D** contains proper divisors of zero. Consequently, **D is not an integral domain.**

The relation $(0,1)\cdot(0,1)=(0,0)$ in **D** has an immediate interpretation in terms of differentiation: For $f(x)=x\cdot x=x^2, f(0)=f'(0)=0$.

More general functions

An arbitrary differentiable function f: $\mathbf{R}\to\mathbf{R}$ can be extended to the function f: $\mathbf{D}\to\mathbf{D}$ by means of the formula

$$f((u,u'))=(f(u),u'\cdot f'(u)), \tag{10}$$

where $f'(u)=\frac{d}{du}f(u)$. For example,

$$e^U=e^{(u,u')}=(e^u,u'\cdot e^u),$$
$$\ln U=\ln(u,u')=(\ln u,u'/u),$$
$$\sin U=\sin(u,u')=(\sin u,u'\cdot\cos u),$$

$$\tan^{-1}U=\tan^{-1}(u,u')=\left(\tan^{-1}u,\,u'/(1+u^2)\right),$$

and so on.

It is easy to verify from (10) that the rules (5)–(8) of differentiation arithmetic are equivalent to the usual rules for finding the derivatives of sums, differences, products, and quotients of functions. Also, for the composition $f\circ g$ of differentiable real functions f,g defined by $f\circ g(x)=f(g(x))$, one has

$$\begin{aligned} f\circ g((u,u')) &= (f\circ g(u),\,u'\cdot(f\circ g)'(u)) \\ &= (f(g(u)),\,u'\cdot g'(u)\cdot f'(g(u))) \\ &= f(g(u,u')), \end{aligned} \tag{11}$$

by the chain rule.

Since the evaluation of a formula can be considered to be a composition of arithmetic operations and previously defined functions, (11) enables the use of automatic differentiation for the types of functions encountered in elementary analysis [7].

There is an immediate identification of elements $U=(u,u')$ of $\mathbf{D}$ with points of the real plane $\mathbf{R}^2$. A differentiable real function f: $\mathbf{R}\to\mathbf{R}$ can be extended to a function from $\mathbf{R}$ into $\mathbf{D}$ by means of the mapping $x\mapsto(f(x),f'(x))$. (It would be ambiguous to denote this function by f.) For example, $x\mapsto(x,1)$, so the graph of the identity function is the line $u'=1$ in a (u,u')-coordinate system for $\mathbf{D}$. Similarly, a constant function with value c is represented by the point $(c,0)$ on the u-axis. As x ranges over the real numbers $\mathbf{R}$, the set of points $(f(x),f'(x))$ forms a curve F which is called the phase plane diagram of f. This curve can reveal some interesting facts about f. For example, the points at which F crosses the u-axis $(f'(x)=0)$ are critical points of f. Similarly, zeros of f are simply points at which F crosses the u'-axis $(f(x)=0)$. In certain cases, it is possible to express F explicitly in the (u,u')-coordinate system for $\mathbf{D}$. For example, if $f(x)=x^2+4$, then F is the graph of the function $u=g(u')$ defined by

$$u=\tfrac{1}{4}(u')^2+4$$

in this geometry. It follows that the point (4,0) corresponds to a critical point $\hat{x}$ of f, at which f has the value $f(\hat{x})=4$. It can also be deduced that $f(\hat{x})=4$ is a global minimum of f, because all the rest of F lies to the right of (4,0), that is, $f(x)\geqslant 4$ for all x.

Another formulation of differentiation arithmetic

Ordered-pair arithmetics are very convenient if a computer is to be used, since real and imaginary parts of complex numbers, for example, are usually stored in different locations, and are calculated separately in the course of an evaluation of a complex function. However, for hand computation, the notation

$$z=x+iy \tag{12}$$

is often preferred for the complex number $z=(x,y)$, where i is the "imaginary unit" which keeps the real and imaginary parts of z distinct. In order to compute with imaginary numbers in the form (12) rules are prescribed for dealing with i, the most important of which are

$$0\cdot i=i\cdot 0=0 \quad\text{and}\quad i^2=-1. \tag{13}$$

A similar formulation of differentiation arithmetic can be made, using ε in a similar way to i in complex arithmetic. An element $U=(u,u')\in\mathbf{D}$ can be written

$$U=u+\varepsilon u',$$

and the rules corresponding to (13) are:

$$0\cdot\varepsilon=\varepsilon\cdot 0=0 \quad\text{and}\quad \varepsilon^2=0. \tag{14}$$

The second of these is equivalent to (9). Addition and subtraction have the obvious rules, while for multiplication,

$$\begin{aligned}(u+\varepsilon u')\cdot(v+\varepsilon v') &= uv + u\cdot\varepsilon v' + \varepsilon u'\cdot v + \varepsilon^2\cdot u'v' \\ &= uv + \varepsilon(uv' + u'v),\end{aligned} \tag{15}$$

which is simply (7). The same reasoning can be used in the case of division: If $v \neq 0$, then $V = v + \varepsilon v'$ has the reciprocal

$$\begin{aligned}V^{-1} = (v+\varepsilon v')^{-1} &= \frac{1}{v+\varepsilon v'}\cdot\frac{v-\varepsilon v'}{v-\varepsilon v'} \\ &= \frac{v-\varepsilon v'}{v^2} = \frac{1}{v} - \varepsilon\frac{v'}{v^2},\end{aligned} \tag{16}$$

so that for $U/V = U\cdot V^{-1}$, the formula equivalent to (8) follows from (15) and (16).

Because of the condition $\varepsilon^2 = 0$, it seems natural to call ε an **infinitesimal unit**. However, there is no suggestion here that ε is somehow "small", or comparable to real numbers in any way. Basically, ε is a marker to keep the components of an ordered-pair variable separate from each other. The use of "infinitesimals" here to *evaluate* derivatives, using an arithmetic based on (4), is quite different from using infinitesimals to *derive* formulas for derivatives, which requires quite a bit more justification [**5**].

The equivalent of formula (10) in this notation is

$$f((x,h)) = f(x) + \varepsilon h f'(x),$$

which can be considered to be the result of discarding "higher order terms" in the Taylor series expansion of $f(x+\varepsilon h)$, where $\varepsilon \in \mathbf{R}$ *is* considered to be small in this case. This method of approximation of functions by linearization for small perturbations has a long history in applied mathematics. The formulation of differentiation arithmetic in terms of ordered pairs, however, appears to be easier to implement on a computer than the more traditional approach of this section.

Extensions of differentiation arithmetic

Differentiation arithmetic can be generalized to (i) higher derivatives, and (ii) several independent variables. For example, first and second derivatives as well as function values can be calculated by an arithmetic based on ordered triples,

$$U = (u, u', u''),$$

using again well-known rules for obtaining the first and second derivatives of sums, differences, products, quotients, etc. In this arithmetic, the independent variable for differentiation is represented by $X = (x,1,0)$, and constants by $C = (c,0,0)$. Leibniz' rule is very useful for the derivation of the rules for multiplication and division. This idea can be extended to obtain derivatives of arbitrary order, or alternatively Taylor coefficients of functions, since recurrence relations are known for formation of the $(n+1)$-tuples

$$F(X) = (f_0, f_1, \ldots, f_n),$$

where $X = (x, h, 0, \ldots, 0)$, and

$$f_k = \frac{h^k}{k!}f^{(k)}(x), \qquad k = 0,1,2,\ldots,n, \tag{17}$$

[**6**], [**7**]. In (17) the usual conventions $f^{(0)}(x) = f(x)$ and $0! = 1$ are observed. Values of derivatives can be obtained from (17) by setting $h = 1$ and multiplying by the appropriate factorials. As in the last section, the representation

$$F(X) = (f_0, f_1, \ldots, f_n) = f_0 + \varepsilon f_1 + \cdots + \varepsilon^n f_n \tag{18}$$

can be adopted, with $\varepsilon^{n+1}=0$. In theory (but not in practice), this can be extended to formal power series,

$$F(X)=f_0+\varepsilon f_1+\cdots+\varepsilon^n f_n+\cdots.$$

It is interesting to note that the extensions of the arithmetic rules (5)–(8) to formal power series results in an integral domain [**4**], which is not true for the finite segments (18).

Another way to extend differentiation arithmetic is to functions of several variables, that is, to the calculation of partial derivatives. In this case, one works with

$$F(X)=(f(x),\nabla f(x)),$$

where $\nabla f(x)$ denotes the gradient vector of f at $x=(x_1,x_2,\dots,x_n)$, that is

$$\nabla f(x)=\left(\frac{\partial f(x)}{\partial x_1},\frac{\partial f(x)}{\partial x_2},\dots,\frac{\partial f(x)}{\partial x_n}\right).$$

In this case, the ith independent variable is represented by $X_i=(x_i,\mathbf{e}_i)$, where $\mathbf{e}_i$ is the ith unit vector, and constants by $C=(c,\mathbf{0})$, where $\mathbf{0}$ denotes the zero vector $\mathbf{0}=(0,0,\dots,0)$. Each component of $\nabla f(x)$ is calculated according to exactly the same rules given above for the various arithmetic operations, and formula (10) applies componentwise. The differentiation arithmetic presented in this paper is actually the special case of this **gradient arithmetic** for $n=1$ [**8**].

Differentiation arithmetic has been extended to higher partial derivative and Taylor coefficients in several variables in the same way as for functions of a single variable. Second partial differentiation is based on the representation

$$F(X)=(f(x),\nabla f(x),Hf(x)),$$

where $Hf(x)$ denotes the Hessian matrix

$$Hf(x)=\left(\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j}\right)$$

of second partial derivatives of f at x.

Increasing the order of derivatives needed or the number of independent variables quickly puts automatic differentiation out of reach of hand calculation. However, the rules of automatic differentiation are easy to program, and are well-suited to parallel computation, since, for example, each component of the gradient vector is calculated according to exactly the same rule as for the others. The results have the accuracy of symbolic differentiation, but without the need for a lot of extra program code. Furthermore, differentiation arithmetic can be combined with other forms of arithmetic, such as interval arithmetic, to provide information about the values of functions and their derivatives over ranges of values of the variables [**6**], [**7**].

A simple application of differentiation arithmetic

The use of Newton's method to solve an equation of the form

$$f(x)=0 \tag{19}$$

requires values $f(x_n)$ and $f'(x_n)$ of the function and its derivative for each approximation x_n to a solution $x=x^*$ of (19). These values are given directly by evaluation of the function f using differentiation arithmetic. For

$$X_n=(x_n,1),$$

differentiation arithmetic gives

$$F_n=F(X_n)=(f_n,f_n')=(f(x_n),f'(x_n)),$$

and thus for

$$x_{n+1} = x_n - f_n/f_n',$$

the next approximation is

$$X_{n+1} = (x_{n+1}, 1).$$

Starting with $x_0 = -2.1$, the results for the function (1) were:

$$x_0 = -2.100000000, \quad F(X_0) = (27.90000000, 301.0000000),$$
$$x_1 = -2.192691030, \quad F(X_1) = (13.37627448, 81.79756235),$$
$$x_2 = -2.356220041, \quad F(X_2) = (6.065540821, 24.64201868),$$
$$x_3 = -2.602366304, \quad F(X_3) = (2.377991960, 9.267989479),$$
$$x_4 = -2.858947515, \quad F(X_4) = (0.633698948, 5.066193105),$$
$$x_5 = -2.984031367, \quad F(X_5) = (0.064651938, 4.098156631),$$
$$x_6 = -2.999807225, \quad F(X_6) = (0.000771212, 4.001156985),$$
$$x_7 = -2.999999972, \quad F(X_7) = (0.000000112, 4.000000168),$$
$$x_8 = -3.000000000, \quad F(X_8) = (0.000000000, 4.000000000).$$

Thus, $f(-3) = 0, f'(-3) = 4$, and $x^* = -3$ is a solution of (19). These results were computed using an HP15-C programmable hand calculator with subroutines for differentiation arithmetic used in place of the corresponding arithmetic operations.

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Tiling with Incomparable Cuboids

CHARLES H. JEPSEN
Grinnell College
Grinnell, IA 50112

In this note we answer three questions raised by Richard K. Guy in his column Unsolved Problems in *The American Mathematical Monthly* of December 1984 [1]. Guy discusses tiling a three-dimensional cuboid using (at least two) incomparable cuboids with integer sides. Two cuboids are *incomparable* if neither will fit inside the other with sides parallel. (To compare two cuboids, write their dimensions in increasing order: $a_1 \leqslant a_2 \leqslant a_3$, $b_1 \leqslant b_2 \leqslant b_3$; the cuboids are incomparable if there is an i and a j such that $a_i < b_i$ and $a_j > b_j$.) Bill Sands found a tiling of a $3 \times 4 \times 15$ cuboid (volume 180) using the six pieces $1 \times 1 \times 15$, $1 \times 2 \times 11$, $1 \times 3 \times 10$, $2 \times 3 \times 6$, $2 \times 4 \times 4$, $3 \times 3 \times 5$. (See FIGURE 1).

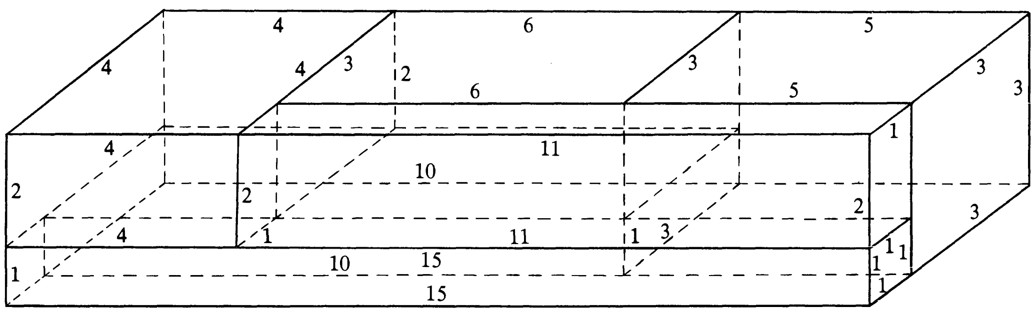


FIGURE 1.

Guy poses the following question:

1. Is there a smaller integer tiling of the cuboid than that of Sands?

In the special case where the cuboid to be tiled is a cube, Guy poses the following problems:

2. Give a neat proof that there is no 6-piece incomparable tiling of a cube.

3. Is there a nontrivial 7-piece incomparable tiling of a cube? (By nontrivial we mean that not all pieces have the edge of the cube as a dimension. A trivial 7-piece tiling of an $n \times n \times n$ cube can be obtained by erecting cuboids of height n over 7 incomparable rectangles that tile the $n \times n$ square base.)

The answer to question 1 is yes. FIGURE 2 shows a $3 \times 5 \times 9$ cuboid (volume 135) tiled with the

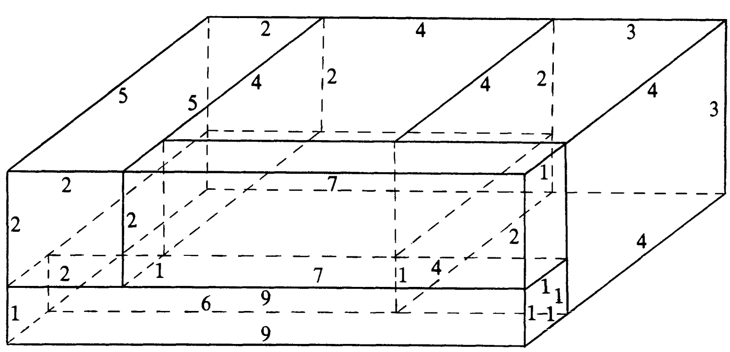


FIGURE 2.

six pieces $1 \times 1 \times 9$, $1 \times 2 \times 7$, $1 \times 4 \times 6$, $2 \times 2 \times 5$, $2 \times 4 \times 4$, $3 \times 3 \times 4$. Further, we prove (Theorem 6 below) that our example is the cuboid of minimal volume that admits an incomparable tiling.

The answer to question 3 is also yes. In FIGURE 3 we exhibit a $10 \times 10 \times 10$ cube tiled with the

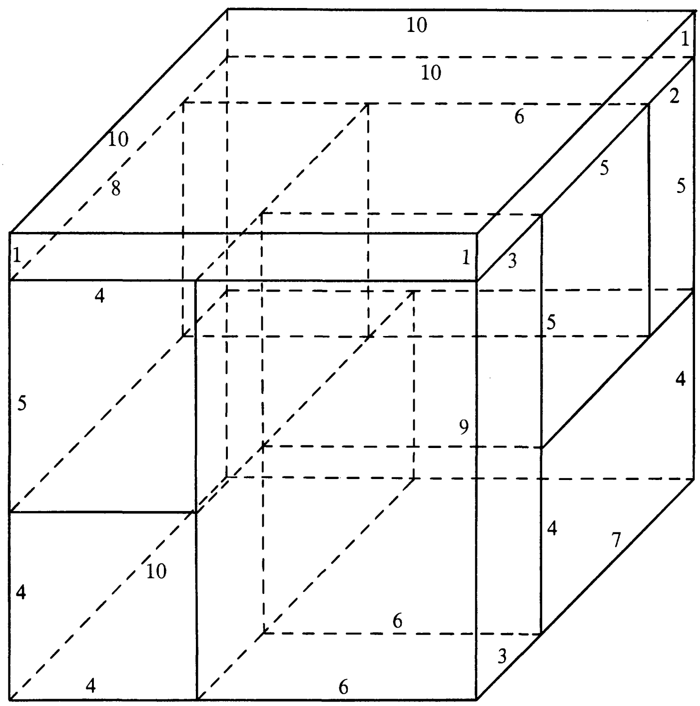


FIGURE 3.

seven pieces $1 \times 10 \times 10$, $2 \times 5 \times 10$, $3 \times 6 \times 9$, $4 \times 4 \times 10$, $4 \times 5 \times 8$, $4 \times 6 \times 7$, $5 \times 5 \times 6$; FIGURE 4 gives the variation of this tiling obtained by replacing pieces $1 \times 10 \times 10$ and $2 \times 5 \times 10$ by pieces $1 \times 8 \times 10$ and $2 \times 6 \times 10$. It is not known if this example is the cube of minimal volume that can be tiled. (See questions 1 and 2 posed at the end of the paper.)

Finally, we handle problem 2 by establishing the impossibility of a 6-piece tiling of a cube as a consequence of our general tiling results (see the Corollary to Theorem 3).

Note that the tiling of FIGURE 2 has the same pattern as FIGURE 1. Indeed, we derived this example by looking for a tiling of a $3 \times 5 \times n$ cuboid using the pattern of FIGURE 1, hoping to find one with n less than 12. First place a $1 \times 1 \times n$ piece as shown. A $1 \times 2 \times (n-2)$ piece can be placed on top of the $1 \times 1 \times n$ with just enough room for a $2 \times 2 \times 5$ as shown. The remainder of the $3 \times 5 \times n$ cuboid is then tiled with pieces $1 \times 4 \times a$ (under the $2 \times 2 \times 5$), $2 \times 4 \times b$ (over the $1 \times 4 \times a$), and $3 \times 4 \times c$ (in the corner), where a, b, c satisfy the equations $a = b + 2$ and $n = a + c$. In order that $2 \times 2 \times 5$, $2 \times 4 \times b$ and $3 \times 4 \times c$ be incomparable, we need $5 > b > c \geqslant 3$. This forces $b = 4$ and $c = 3$. But then $a = 6$ and $n = 9$, giving a 6-piece tiling of the $3 \times 5 \times 9$ cuboid in FIGURE 2.

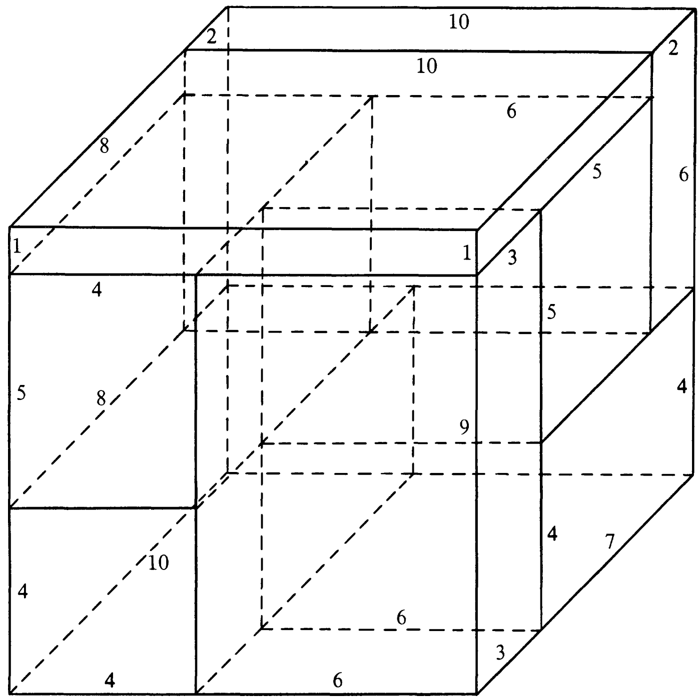


FIGURE 4.

In Theorem 2 we show that *all* 6-piece tilings exhibit the pattern of FIGURES 1 and 2. Theorem 1 gives the minimal number of pieces in a tiling. Before proceeding, we remark that our results in three dimensions have known counterparts in two dimensions: an incomparable tiling of a rectangle requires at least 7 pieces, and the rectangle of minimal area that can be tiled has dimensions 13×22 and area 286. (See [**2**] for these and other results.)

THEOREM 1. *An incomparable tiling of a cuboid requires at least* 6 *pieces.*

Proof. If at most 4 pieces tile a cuboid, we see easily that not all pairs of pieces are incomparable. Suppose 5 incomparable pieces tile a cuboid. A straightforward check shows that 3 pieces occupy two corners each, 2 pieces occupy one corner each, and the pieces are placed as in FIGURE 5. Note that $a_1 > b_1$ and $a_2 = b_2$, $d_2 > b_2$ and $d_3 = b_3$, $c_3 > b_3$ and $c_1 = b_1$. For the

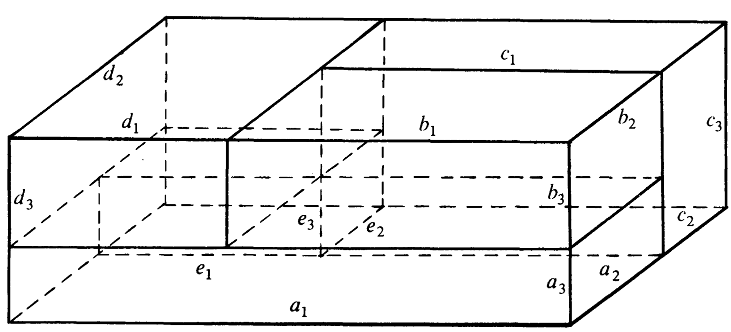


FIGURE 5.

pieces to be incomparable, we must have $a_3 < b_3$, $d_1 < b_1$, $c_2 < b_2$. But then $b_1 > d_1 = e_1$, $b_2 > c_2 = e_2$, $b_3 > a_3 = e_3$. Thus the pieces $b_1 \times b_2 \times b_3$ and $e_1 \times e_2 \times e_3$ are not incomparable, a contradiction.

THEOREM 2. *A 6-piece incomparable tiling of a cuboid has its pieces arranged in the pattern of* FIGURE 1.

Proof. Consider a cuboid tiled with 6 incomparable pieces and make three observations.

a) No face of the cuboid has 5 or more pieces appearing on it.

If 6 pieces appear on a face, the result is a trivial tiling with this face as base, impossible since an incomparable tiling of a rectangle requires at least 7 pieces. If 5 pieces appear on a face, the sixth piece lies at the end of a collection of these 5 pieces. This collection would then form a trivial tiling of a cuboid, again impossible.

b) No face of the cuboid has its 4 corners occupied by 4 different pieces.

Suppose pieces (1) through (4) each occupy one corner of a face as in FIGURE 6a and call a dimension perpendicular to this face a length. Since two more pieces complete the tiling, we check that: i) a pair of horizontally adjacent pieces, say (1) and (2), have length less than that of the cuboid; ii) these two pieces have equal length; iii) the remaining pair, say (3) and (4), have length equal to that of the cuboid. (See FIGURE 6b; the details are omitted.) It follows that either (1) and (2) or (3) and (4) are not incomparable (an easy check).

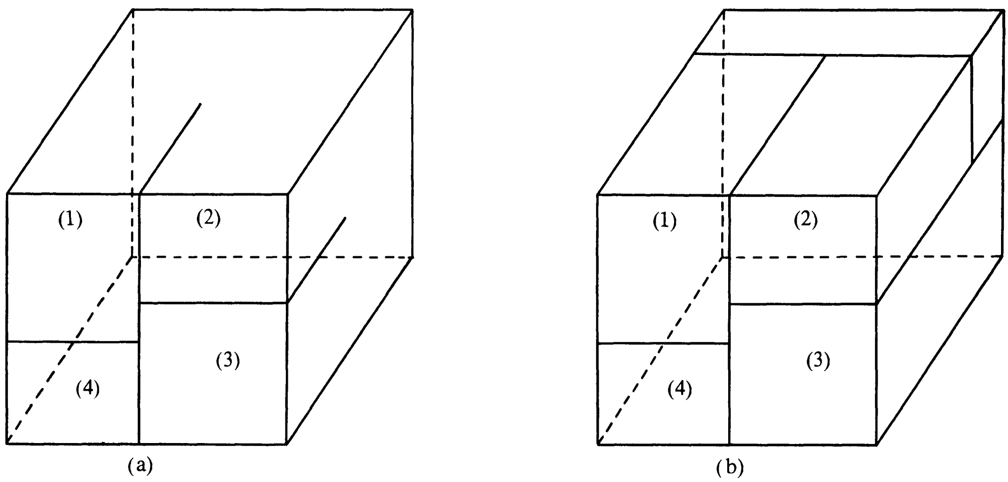


FIGURE 6.

c) No face of the cuboid has its 4 corners occupied by only 2 different pieces.

If such a face exists, we conclude from b) that the tiling contains four pieces as shown in FIGURE 7a. We may assume $a > b$ so piece (2) does not touch either (3) or (4). But then the face containing pieces (2)–(4) (FIGURE 7b) has at least 5 pieces appearing on it, contradicting a).

From b) and c) we see that the tiling has three pieces arranged as in FIGURE 8a. We show this leads to the pattern of FIGURE 1.

If two pieces in FIGURE 8a fail to touch each other, there is a cuboid-shaped hole between them. Such a hole cannot be filled by a piece used to occupy one of the two remaining corners of the cuboid. Moreover, no two holes can be filled by a single piece. Since the tiling is completed using three more pieces with two in the corners, exactly one pair of pieces in FIGURE 8a do not touch; the result is FIGURE 8b. Adding three pieces, we obtain the 6-piece tiling in FIGURE 8c whose pattern is that of FIGURE 1. This being the only possible tiling, the result is established.

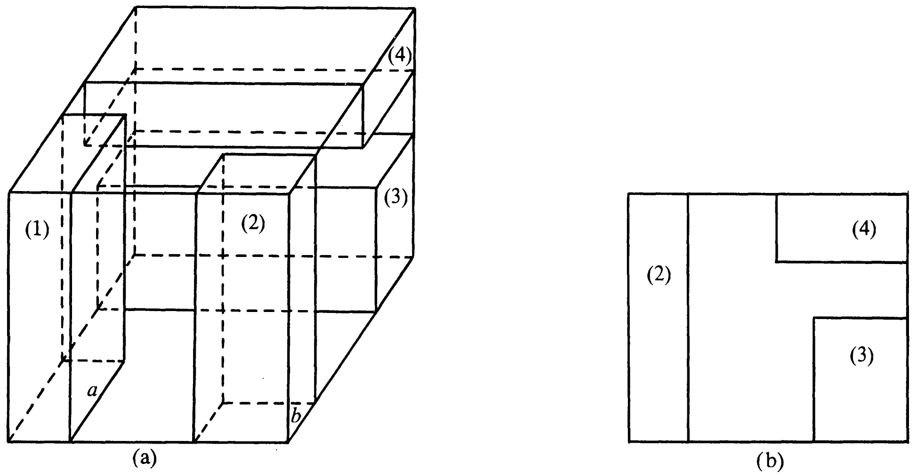


FIGURE 7.

Before establishing further results about cuboids, we combine Theorem 2 with the following observation about cubes to resolve Guy's second problem.

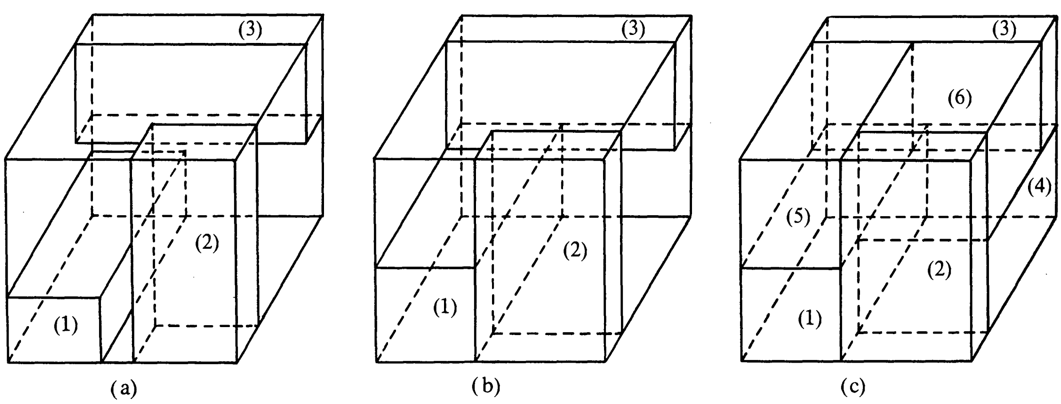


FIGURE 8.

THEOREM 3. *The pattern of* FIGURE 1 *cannot be used to produce a* 6*-piece incomparable tiling of a cube.*

Proof. This result is not new (for example, it is referred to in [**1**]); we give a proof for the sake of completeness. Suppose an $n\times n\times n$ cube is tiled with 6 pieces as labeled in FIGURE 9; the dimensions of the pieces are: (1) $a\times b\times n$, (2) $c\times(n-b)\times n$, (3) $d\times(n-a)\times n$, (4) $a\times(n-b)\times(n-c)$, (5) $b\times(n-a)\times(n-d)$, (6) $(n-a)\times(n-b)\times(n-c-d)$. Assume $a\leqslant b$

so $n-a \geqslant n-b$. For pieces (2) and (3) to be incomparable, we need $c > d$. But then $n-c < n-d$ and pieces (4) and (5) are not incomparable.

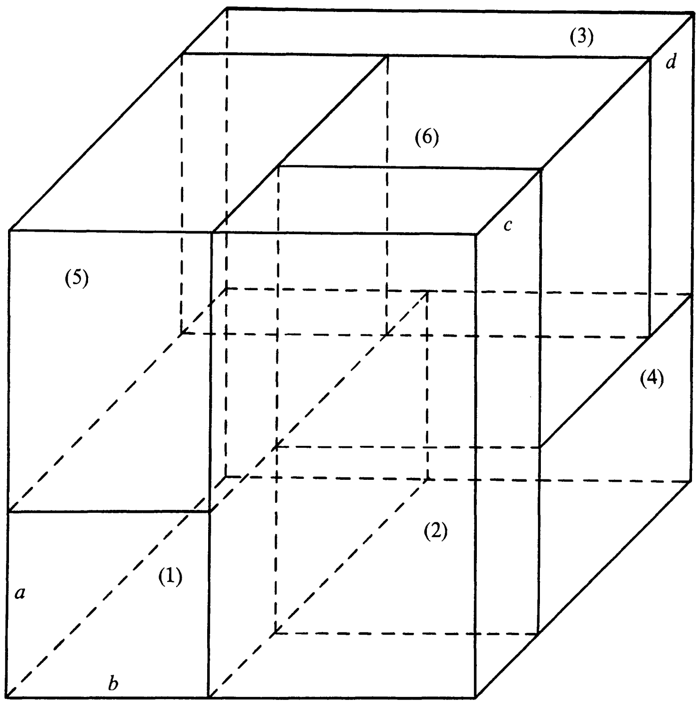


FIGURE 9.

COROLLARY. *There is no 6-piece incomparable tiling of a cube.*

Proof. Immediate from Theorems 2 and 3.

To complete our discussion of cubes, we return to the 7-piece tiling of FIGURE 3. This example was derived by finding a 6-piece tiling of an $(n-1) \times n \times n$ cuboid, where each piece has smallest dimension at least 2, and adding a $1 \times n \times n$ piece on top. We show this procedure requires $n \geqslant 10$; hence FIGURE 3 is the cube of minimal volume that admits such a 7-piece tiling.

Consider an $(n-1) \times n \times n$ cuboid tiled with 6 pieces as labeled in FIGURE 9; the pieces now have dimensions: (1) $a \times b \times n$, (2) $c \times (n-b) \times (n-1)$, (3) $d \times (n-1-a) \times n$, (4) $a \times (n-b) \times (n-c)$, (5) $b \times (n-1-a) \times (n-d)$, (6) $(n-1-a) \times (n-b) \times (n-c-d)$. Incomparability of pieces (2) and (3) says at least one of the following inequalities must hold: $d < c$, $b \leqslant a$. But incomparability of (4) and (5) says that either of these inequalities implies the other. Hence both are satisfied. Incomparability of (2) and (5) then yields $c < b$. Thus $d < c < b \leqslant a$; since $d \geqslant 2$, $a \geqslant 4$. For (1) and (5) to be incomparable, we need $n-1-a > a$, i.e., $n \geqslant 2a+2$. Hence $n \geqslant 10$. Letting $n = 10$ (i.e., $a = b = 4$, $c = 3$, $d = 2$), we get the example in FIGURE 3.

We resume our investigation of cuboids by looking at cases where one or two dimensions are small. Throughout the following discussion, the dimensions are assumed in increasing order: an $A \times B \times C$ cuboid has $A \leqslant B \leqslant C$.

Consider first $2 \times B \times C$ cuboids. Tilings of such cuboids seem to require both a substantial number of pieces and large values of B and C. (See questions 3 and 4 posed at the end of the paper.) A trivial tiling with all pieces of height 2 requires 7 pieces and forces the volume of the cuboid to be at least $2 \cdot 13 \cdot 22 = 572$. At the opposite extreme, if all pieces have height 1, a minimum of 14 pieces are needed (7 per layer), and the volume of the cuboid again must be at least 572 and likely is far greater. Finally, suppose there are pieces of height 2 but the tiling is nontrivial. After the pieces of height 2 are placed in the cuboid, the remaining portion of the $B \times C$ base is covered with two layers of incomparable pieces of height 1. This requires a minimum of 6 pieces (3 per layer) and probably many more. But the 6 such pieces of smallest total volume are $1 \times 1 \times 11$, $1 \times 2 \times 10$, $1 \times 3 \times 9$, $1 \times 4 \times 8$, $1 \times 5 \times 7$, $1 \times 6 \times 6$; these pieces have volume 161, forcing the volume of the cuboid to be even larger.

Now consider cuboids with smallest dimension 3.

THEOREM 4. *There is no incomparable tiling of a* $3 \times 3 \times C$ *cuboid.*

Proof. The only collection of 6 or more incomparable pieces that would fit inside such a cuboid is $1 \times 1 \times n$, $1 \times 2 \times p$, $1 \times 3 \times q$, $2 \times 2 \times r$, $2 \times 3 \times s$, $3 \times 3 \times t$. But removing the last piece would result in a $3 \times 3 \times (C - t)$ cuboid to be tiled with 5 pieces, impossible from Theorem 1.

THEOREM 5. *An incomparable tiling of a* $3 \times 4 \times C$ *cuboid consists of* 6 *pieces* (*arranged in the pattern of* FIGURE 1). *Further, the existence of a tiling of such a cuboid requires* $C \geqslant 15$.

Proof. The pieces that tile a $3 \times 4 \times C$ cuboid come from the collection $1 \times 1 \times n$, $1 \times 2 \times p$, $1 \times 3 \times q$, $1 \times 4 \times r$, $2 \times 2 \times s$, $2 \times 3 \times t$, $2 \times 4 \times u$, $3 \times 3 \times v$. We show that the pieces $1 \times 4 \times r$ and $2 \times 2 \times s$ cannot appear.

Suppose a $1 \times 4 \times r$ is placed in the $3 \times 4 \times C$ cuboid. This piece must lie horizontally in either the top or bottom layer; assume it lies at the bottom, creating a $2 \times 4 \times r$ hole above it. Consider the $1 \times 3 \times 4$ cross sections that intersect the $1 \times 4 \times r$; some have the $1 \times 2 \times 4$ hole filled with pieces $1 \times 2 \times p$ and $2 \times 3 \times t$ (FIGURE 10a). That is, pieces $1 \times 2 \times p$ and $2 \times 3 \times t$ are placed above the $1 \times 4 \times r$. Since p is greater than r and t, there exists a $1 \times 3 \times 4$ cross section that intersects only the $1 \times 2 \times p$ (FIGURE 10b). The remaining portion of this cross section must be filled by pieces $1 \times 1 \times n$ and $3 \times 3 \times v$. Placement of these pieces creates a cross section

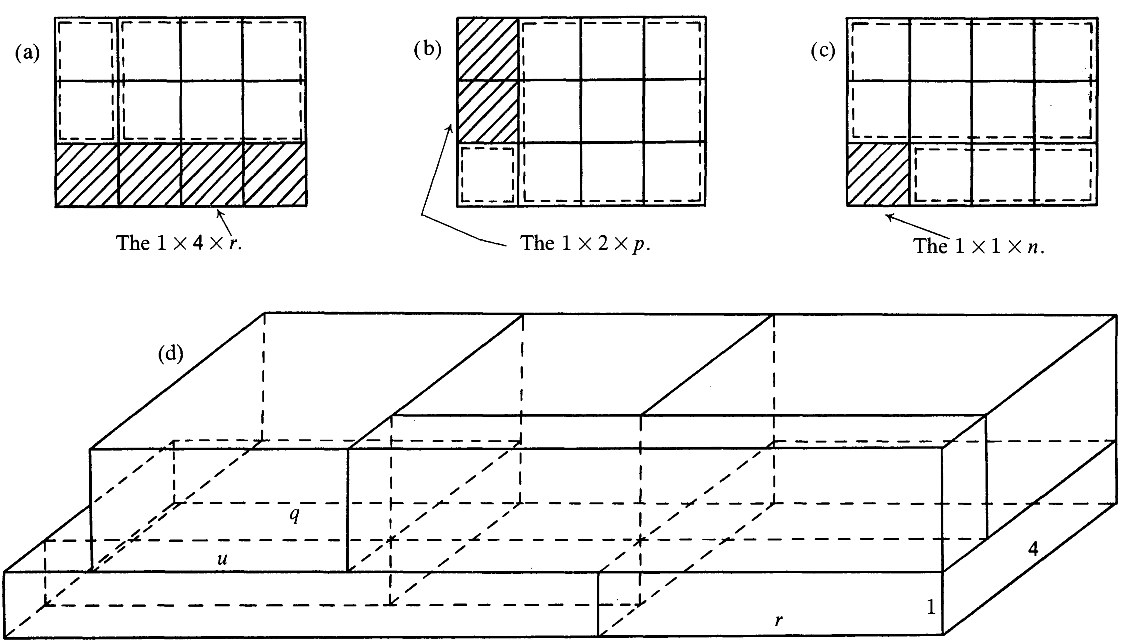


FIGURE 10.

intersecting only the $1 \times 1 \times n$ (FIGURE 10c). The hole in this cross section is filled by the $1 \times 3 \times q$ and $2 \times 4 \times u$ (the $2 \times 2 \times s$ will not work because $s > t > v \geqslant 3$ forces $s \geqslant 5$). Since $q > u$, these 7 pieces do not complete a cuboid (see FIGURE 10d), and the remaining piece (the $2 \times 2 \times s$) is of no help. We have reached an impossibility, proving that the piece $1 \times 4 \times r$ is not present.

Consider now the piece $2 \times 2 \times s$; we check easily that $s \geqslant 5$. But then placement of this piece creates cross sections that can be filled only by the $1 \times 2 \times p$ and $2 \times 3 \times t$, impossible because $s > t$. (The details are omitted.)

It follows that an incomparable tiling of a $3 \times 4 \times C$ cuboid uses the 6 pieces $1 \times 1 \times n$, $1 \times 2 \times p$, $1 \times 3 \times q$, $2 \times 3 \times t$, $2 \times 4 \times u$, $3 \times 3 \times v$ where $n > p > q > t > u \geqslant 4$ and $t > v \geqslant 3$. Further, the 6 pieces are arranged as in FIGURE 9. We show this forces $C \geqslant 15$. From the pattern of the tiling in FIGURE 9, we see that: the piece $3 \times 3 \times v$ occupies two corners of the $3 \times 4 \times C$ cuboid, and the $1 \times 1 \times n$ and $1 \times 2 \times p$ are stacked on each other beside the $3 \times 3 \times v$. Hence we may assume the $3 \times 3 \times v$ is piece number (2), the $1 \times 1 \times n$ is (1), and the $1 \times 2 \times p$ is (5). We then find that the $2 \times 4 \times u$ is (3), the $1 \times 3 \times q$ is (4), and the $2 \times 3 \times t$ is (6). It follows that $C = n = u + p = q + v = u + t + v$. Since $u \geqslant 4$ and $p > q$, $n \geqslant p + 4 > q + 4$; thus $v = n - q > 4$. Hence $v \geqslant 5$, $t \geqslant 6$ and $C = u + t + v \geqslant 4 + 6 + 5 = 15$.

We are now in a position to prove the minimality of FIGURE 2.

THEOREM 6. *The* $3 \times 5 \times 9$ *cuboid* (*volume* 135) *in* FIGURE 2 *is the cuboid of minimal volume that admits an incomparable tiling.*

Proof. Suppose there is an $A \times B \times C$ cuboid of volume less than 135 that can be tiled; our work above shows that $A \geqslant 3$ and $A + B \geqslant 8$. Note that the largest cuboid with these restrictions has dimensions $4 \times 4 \times 8$ and volume 128.

We show that a tiling of such a cuboid has exactly 6 pieces. There cannot be as many as 8 pieces; indeed, the smallest volume that 8 pieces can occupy is $1 \times 1 \times 9 + 1 \times 2 \times 8 + 1 \times 3 \times 7 + 1 \times 4 \times 6 + 1 \times 5 \times 5 + 2 \times 2 \times 5 + 2 \times 3 \times 4 + 3 \times 3 \times 3 = 166$. Consider possible 7-piece tilings. If at most 3 pieces have side 1, the total volume is at least 153 (the collection of smallest volume being $1 \times 1 \times 8, 1 \times 2 \times 7, 1 \times 3 \times 6, 2 \times 2 \times 6, 2 \times 3 \times 5, 2 \times 4 \times 4, 3 \times 3 \times 3$). A similar check shows that 5 or more pieces of side 1 yield a total volume of at least 129. It follows that exactly 4 pieces have side 1. Further, we find that the only possible collection is $1 \times 1 \times 8$, $1 \times 2 \times 7$, $1 \times 3 \times 5$, $1 \times 4 \times 4$, $2 \times 2 \times 6$, $2 \times 3 \times 4$, $3 \times 3 \times 3$, of volume $128 = 4 \times 4 \times 8$. But these pieces cannot form a tiling because there is no room for both a $2 \times 2 \times 6$ and a $3 \times 3 \times 3$ inside a $4 \times 4 \times 8$ cuboid.

We now look for all possible collections of 6 pieces of total volume $V = A \cdot B \cdot C < 135$ with $A \geqslant 3$, $A + B \geqslant 8$ and where each piece fits inside an $A \times B \times C$ cuboid. A computer search yielded the 25 collections listed in the TABLE below. (This program was written by my colleague John D. Stone; his assistance is gratefully acknowledged.) To prove the theorem, we eliminate these 25 possibilities.

In all but the three cases (2), (12) and (15), we have $B < C - 1$ and a piece of side $C - 1$. Placement of this piece creates a $1 \times A \times B$ hole at its end. This hole cannot be filled by a $1 \times A \times B$ piece (for its removal would leave an $A \times B \times (C - 1)$ cuboid to be tiled with 5 pieces), and there is no smaller piece. A slight refinement of this argument disposes of the three remaining cases. (In (15), there is no piece to fit inside the $2 \times 4 \times 4$ hole at the end of the $1 \times 2 \times 6$.)

Since all cases with volume less than 135 have been eliminated, the result is established.

We finish by posing four questions, two that arise in tiling a cube and two that arise in tiling a $2 \times B \times C$ cuboid.

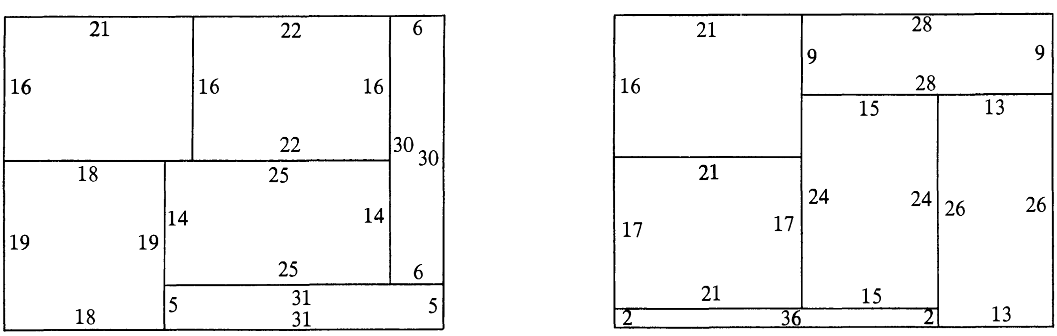
1. Is there a pattern other than those of FIGURES 3 and 4 that will produce a 7-piece incomparable tiling of a cube? Added in proof: The answer is yes. I recently found a tiling of an $11 \times 11 \times 11$ cube with a different pattern using the seven pieces $1 \times 7 \times 11, 2 \times 8 \times 10, 3 \times 4 \times 11$, $3 \times 7 \times 10$, $3 \times 8 \times 9$, $4 \times 8 \times 8$, $5 \times 7 \times 8$.

	The pieces						The cuboid
(1)	1 × 1 × 7,	1 × 2 × 6,	1 × 3 × 5,	2 × 2 × 5,	2 × 3 × 4;	3 × 3 × 3	$V = 105 = 3 \times 5 \times 7$
(2)	1 × 1 × 6,	1 × 3 × 5,	1 × 4 × 4,	2 × 2 × 5,	2 × 3 × 4,	3 × 3 × 3	$V = 108 = 3 \times 6 \times 6$
(3)	1 × 1 × 7,	1 × 3 × 6,	1 × 4 × 4,	2 × 2 × 5,	2 × 3 × 4,	3 × 3 × 3	$V = 112 = 4 \times 4 \times 7$
(4)	1 × 1 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 2 × 6,	2 × 3 × 4,	3 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(5)	1 × 1 × 8,	1 × 2 × 7,	1 × 3 × 5,	2 × 2 × 6,	2 × 4 × 4,	3 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(6)	1 × 1 × 8,	1 × 2 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 2 × 6,	3 × 3 × 4	$V = 120 = 3 \times 5 \times 8$
(7)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 4,	2 × 2 × 6,	2 × 3 × 4,	3 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(8)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 5,	2 × 2 × 5,	2 × 3 × 4,	3 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(9)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 6,	1 × 5 × 5,	2 × 2 × 6,	2 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(10)	1 × 2 × 7,	1 × 3 × 5,	1 × 4 × 4,	2 × 2 × 6,	2 × 3 × 4,	3 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(11)	1 × 2 × 8,	1 × 3 × 7,	1 × 4 × 6,	1 × 5 × 5,	2 × 2 × 4,	2 × 3 × 3	$V = 120 = 3 \times 5 \times 8$
(12)	1 × 1 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 2 × 6,	2 × 3 × 5,	3 × 3 × 3	$V = 126 = 3 \times 6 \times 7$
(13)	1 × 1 × 7,	1 × 2 × 6,	1 × 4 × 5,	2 × 3 × 5,	2 × 4 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(14)	1 × 1 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 2 × 6,	2 × 4 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(15)	1 × 1 × 8,	1 × 2 × 6,	1 × 4 × 5,	2 × 2 × 5,	2 × 4 × 4,	3 × 3 × 4	$V = 128 = 4 \times 4 \times 8$
(16)	1 × 1 × 8,	1 × 2 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 2 × 5,	3 × 4 × 4	$V = 128 = 4 \times 4 \times 8$
(17)	1 × 1 × 8,	1 × 2 × 7,	1 × 3 × 6,	1 × 4 × 5,	2 × 4 × 4,	3 × 3 × 4	$V = 128 = 4 \times 4 \times 8$
(18)	1 × 1 × 8,	1 × 2 × 7,	1 × 3 × 6,	2 × 2 × 5,	2 × 4 × 4,	3 × 3 × 4	$V = 128 = 4 \times 4 \times 8$
(19)	1 × 1 × 8,	1 × 2 × 7,	1 × 4 × 4,	2 × 2 × 6,	2 × 3 × 5,	3 × 3 × 4	$V = 128 = 4 \times 4 \times 8$
(20)	1 × 1 × 8,	1 × 2 × 7,	1 × 4 × 5,	2 × 2 × 6,	2 × 3 × 5,	2 × 4 × 4	$V = 128 = 4 \times 4 \times 8$
(21)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 5,	2 × 2 × 5,	2 × 4 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(22)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 5,	2 × 2 × 7,	2 × 3 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(23)	1 × 1 × 8,	1 × 3 × 7,	1 × 4 × 6,	2 × 2 × 6,	2 × 3 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(24)	1 × 2 × 8,	1 × 3 × 7,	1 × 4 × 4,	2 × 2 × 6,	2 × 3 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$
(25)	1 × 2 × 8,	1 × 3 × 7,	1 × 4 × 5,	2 × 2 × 5,	2 × 3 × 4,	3 × 3 × 3	$V = 128 = 4 \times 4 \times 8$

TABLE 1. **The 25 cases with 6 pieces.**

2. Is the $10 \times 10 \times 10$ cube of FIGURE 3 the cube of minimal volume that admits an incomparable tiling?

3. What is the $2 \times B \times C$ cuboid of minimal volume that admits a nontrivial incomparable tiling? (The smallest example I have found is a $2 \times 35 \times 49$ cuboid tiled with the 11 pieces $1 \times 2 \times 36$, $1 \times 5 \times 31$, $1 \times 6 \times 30$, $1 \times 9 \times 28$, $1 \times 13 \times 26$, $1 \times 14 \times 25$, $1 \times 15 \times 24$, $1 \times 16 \times 22$, $1 \times 17 \times 21$, $1 \times 18 \times 19$, $2 \times 16 \times 21$. The two layers are shown in FIGURE 11.)



1st layer

2nd layer

FIGURE 11.

4. What is the $2 \times B \times C$ cuboid of minimal volume that can be tiled with incomparable pieces all having height 1? Such a tiling is equivalent to two tilings of a $B \times C$ rectangle, say using k and m pieces, so that the entire collection of $k+m$ pieces is incomparable. (My best effort is a $2 \times 74 \times 77$ cuboid tiled with two layers of 8 pieces each. See the two layers in FIGURE 12.)

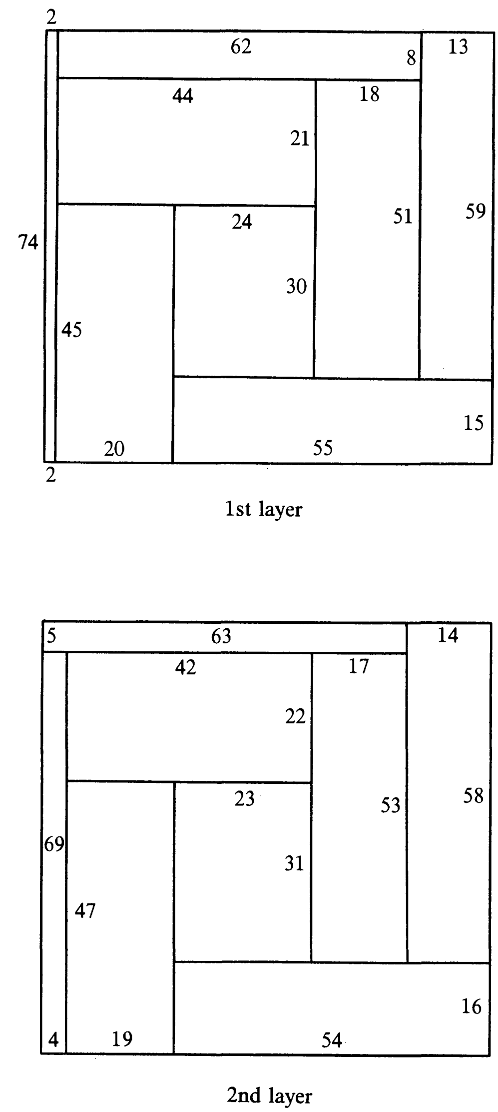


FIGURE 12.

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Special Sets of Smith Numbers

SAMUEL YATES
157 Capri-D, Kings Point
Delray Beach, FL 33445

Wilansky [1] defined a Smith number to be a composite number such as $4 = 2 \times 2$, $22 = 2 \times 11$, or $27 = 3 \times 3 \times 3$, the sum of whose digits is the same as the sum of the digits in its representation as a product of primes. Professor Wilansky's brother-in-law, H. Smith, had a telephone number, $4937775 = 3 \times 5 \times 5 \times 65837$, which is a Smith number, whence the name. Oltikar and Wayland [2] found larger Smith numbers, including one with 362 digits. In this note, I find much larger ones, including one with 7,158 digits. Generally, integers of this size are notoriously difficult to factor, but you will see that if you have large primes of certain forms, it is easy to find large Smith numbers from them.

We can get an idea of the sparseness of Smith numbers from Wilansky's count of them. He found that there are 47 among the first 1,000 nonnegative integers, and between 28 and 42 in each of the next nine 1,000-integer intervals, the average there being 35.

Smith numbers from repunits

A significant result of Oltikar and Wayland's theorems implies that a prime repunit multiplied by a suitable factor produces a Smith number. When they wrote their paper, $R317$, a number written as a string of 317 1's, was the largest known prime repunit, and they gave $2 \times R317 \times 10^{45}$ as the largest Smith number. Later, more than seven years after H. C. Williams and E. Seah first observed that $R1031$ was a probable prime because it satisfied the Fermat congruence when several small bases were used, Williams succeeded in proving that $R1031$ is prime. From it, we obtain the 1,178-digit Smith number $2 \times R1031 \times 10^{147}$. These two repunits also give Smith numbers $27 \times R317 \times 10^{361}$ and $27 \times R1031 \times 10^{1177}$. We can even make a 2,480-digit Smith number that combines both of these repunits into $162 \times R1031 \times R317 \times 10^{1131}$, whose digit sum is 9279.

Similar candidates are going to be hard to find. H. Dubner has determined that there is no prime repunit larger than $R1031$ and smaller than $R10000$ [3].

Smith numbers from doubled primes

A simple technique that produces large Smith numbers, generated from recently discovered very large primes, is given here. It makes use of primes which when doubled become Smith numbers.

Using the notation of [2], $S(n)$ is the sum of the digits of n. Suppose that n is prime; then it is obvious that $2n$ is a Smith number if and only if $S(2n) - S(n) = 2$. To determine what combinations of digits must be present in a prime number so that such a net change will occur, note that $S(2n) - S(n)$ is n for $n = 0, 1, 2, 3$, or 4, and it is $n - 9$ for $n = 5, 6, 7, 8$, or 9. It is also noteworthy that the total change in the digit sum of an integer of any size after doubling is obtained directly by adding the changes in the digit sums of all of its digits. For example, 368 doubled is 736. The sum of the changes for 3, 6, and 8 is $3 - 3 - 1 = -1$. But $S(368) = 17$, $S(736) = 16$, and $16 - 17 = -1$, which is the same as the total digit sum change.

This type of relationship does not hold when multiplying by numbers other than 2 or 5. For example, when $n = 3, 6$, and 8, $S(3n) - S(n)$ is 6, 3, and -2, respectively. So, the total digit sum change of 368 is $6 + 3 - 2 = 7$. However, 368 tripled is 1104, $S(1104) = 6$, $S(368) = 17$, and $6 - 17 = -11$, not equal to 7. The discrepancy is due to the effect of "carries." "Carries" do not affect the total change when a number is doubled or multiplied by 5.

If $a+b=9$, $S(2(a+b))-S(a+b)=0$. Then, the total change after doubling an integer is merely the sum of the changes resulting from those digits which are not equal to 0 or 9 and which cannot be paired into sums equalling 9. For example, when 9027584 is doubled, disregard the 9 and the 0 and the pairs of 9's complements, 2 and 7, and 5 and 4, and the net change becomes -1 because of the 8.

It follows from what was said earlier that a prime doubled becomes a Smith number if the new number has a net change of 2. Then a prime whose digits contain any number of 0's or 9's or pairs of 9-complementary digits, or none of these, with either a 2 left over or two 1's left over, becomes a Smith number when doubled. For instance, prime 281 satisfies this requirement, and $281 \times 2 = 562$. The sum of the digits on both sides of the equality sign is the same, 13, and so 562 is a Smith number.

Largest Smith numbers

Two very large Smith numbers also meet this requirement. They are derived from two titanic primes, primes with 1,000 or more decimal digits [**4**], discovered by H. Dubner in 1984. The primes are the 2,749-digit palindrome $(10^{1375}+999) \times 10^{1373}+1$, and the 1,293-digit palindrome $(10^{647}+999) \times 10^{645}+1$. Each is written as a 1 followed by a long string of zeroes, 999, another long string of zeroes, and a 1. It is easy to see that doubling these primes produces Smith numbers.

Many explorers and discoverers of very large primes find it convenient to generate expressions containing powers of 2, but most of Dubner's have powers of 10. When written in the usual decimal system, his primes are much easier to read and to investigate for suitability as Smith number kernels. The fifth and sixth largest known primes were recently computed by him. They are the two largest known non-Mersenne primes, and serve readily as the cores of what are the two largest known Smith numbers. The largest is a 7,158-digit Smith number obtained from a 7,156-digit prime multiplied by 80. The digit sum on either side of the equality sign is 38. (The large powers of 10 supply long strings of 0's within the numbers.)

$$2 \times 2 \times 2 \times (217833 \times 10^{7150}+1) \times (2 \times 5) = 1742664 \times 10^{7151} + 80.$$

The next is a 7,096-digit Smith number obtained from a 7,094-digit prime by multiplying by 140. The digit sum is 29 on either side of the equality sign.

$$2 \times 7 \times (6006 \times 10^{7090}+1) \times (2 \times 5) = 84084 \times 10^{7091} + 140.$$

(Added in proof: An even larger Smith number with 2,592,699 digits has been found by the author using a technique of McDaniel: $18 \times R1031 \times (138 \times 10^{4071}+1)^{480} \times 10^{636560}$.)

Smith numbers from tetradic primes

Rudolf Ondrejka of Linwood, New Jersey, has been compiling an interesting list of palindromic primes that are called tetradic primes because they are the same primes when viewed "forward, backwards, upside down, or mirror-reflected." I assisted him by computing all the primes which have more than nine digits. Each tetradic prime, except 11, has an odd number of digits, none of which is other than 0, 1, or 8. From his list, which now includes all such primes with 15 digits or less, one can readily select those in which the number of 1's is 2 more than the number of 8's. If the middle digit is 0, and the number of 1's is 1 more than the number of 8's on either side of that zero, that prime doubled is clearly a Smith number. A list of those primes follows.

11	1000180810001	100008101800001	118800101008811
101	1008100018001	100118808811001	118811808118811
1180811	1081810181801	100801000108001	118818101818811
108101801	1100080800011	101801808108101	180011808110081
180101081	1101880881011	108108101801801	180101808101081
10180008101	1110880880111	110000808000011	180118000811081
10810001801	1180000000811	110081808180011	180180101081081
11008080011	1800010100081	110108808801011	180801101108081
11800000811	1808110118081	110881000188011	180811000118081
11881018811	1881100011881	111008808800111	188001101100881
18001010081		111088000880111	188100101001881
		118000000000811	188181101181881

Another subset of tetradic primes that produces Smith numbers is that which contains no 8's only 1's and 0's. If the prime has only two 1's, like 11 or 101, twice the prime is a Smith number. It is also in a subset to which Oltikar and Wayland's Theorem 3 is applicable. In general, if the digit sum of a prime factorization is $7a$ less than the digit sum of the product, where a is a positive integer, multiplying by 10^a produces a Smith number. Smith numbers are obtained from these primes by multiplying

those with 2 1's by 11, those with 5 1's by 3×10,
those with 7 1's by 7×10^5, those with 8 1's by $2 \times 3 \times 10^5$,
those with 10 1's by 5×10^5, and those with 11 1's by $2 \times 7 \times 10^5$.

Here is a list of the tetradic primes with 15 or fewer digits that form Smith numbers in this manner.

$$11 \times 11 = 121 \qquad 101 \times 11 = 1111$$
$$3 \times 100111001 \times (2 \times 5) = 300333003\ 0$$
$$7 \times 110111011 \times (2 \times 5)^5 = 770777077\ 00000$$
$$7 \times 111010111 \times (2 \times 5)^5 = 777070777\ 00000$$
$$7 \times 1100011100011 \times (2 \times 5)^5 = 7700077700077\ 00000$$
$$7 \times 1100101010011 \times (2 \times 5)^5 = 7700707070077\ 00000$$
$$2 \times 3 \times 1101010101011 \times (2 \times 5)^5 = 6606060606066\ 00000$$
$$5 \times 1110110110111 \times (2 \times 5)^5 = 5550550550555\ 00000$$
$$2 \times 7 \times 1110111110111 \times (2 \times 5)^5 = 15541555541554\ 00000$$
$$2 \times 3 \times 10011\ 01010\ 11001 \times (2 \times 5)^5 = 60066\ 06060\ 66006\ 00000$$
$$3 \times 10100\ 00100\ 00101 \times (2 \times 5) = 30300\ 00300\ 00303\ 0$$
$$2 \times 3 \times 10101\ 10001\ 10101 \times (2 \times 5)^5 = 60606\ 60006\ 60606\ 00000$$
$$2 \times 3 \times 10111\ 00000\ 11101 \times (2 \times 5)^5 = 60666\ 00000\ 66606\ 00000$$
$$5 \times 11001\ 11011\ 10011 \times (2 \times 5)^5 = 55005\ 55055\ 50055\ 00000$$
$$2 \times 7 \times 11110\ 01110\ 01111 \times (2 \times 5)^5 = 155540\ 15540\ 15554\ 00000$$

Infinity of Smith numbers

In [5], which proves the existence of an infinity of Smith numbers, it is shown that a Smith number is constructed by multiplying by a suitable number any repunit whose prime factors are known.

Because there is an infinity of repunits, each of which has at least one prime divisor that does not divide any smaller repunit, there is an infinity of Smith numbers. Also, since every prime other than 2 and 5 divides an infinity of repunits, an infinity of Smith numbers can be produced from any given prime.

All but one of the first hundred repunits have now been completely factored, as have over a hundred larger ones. Quite a few others have been factored to the point where their only undetermined factors are probably prime, having satisfied the Fermat congruence in all bases in which they were tested.

Smith numbers from multiples of 3

Because an analytical approach works in constructing Smith numbers by doubling, can a similar reasoning process be used to generate Smith numbers by tripling? It is appropriate beforehand to look at a Smith number that is so generated, but it turns out that there is none.

The only prime divisible by 3 is 3, which when tripled is obviously not a Smith number. A number is divisible by 3 if and only if its digit sum is divisible by 3. If p is a prime other than 3, the product $3p$ is divisible by 3 but the sum $3+p$ is not. Therefore, the sum of the digits in the product cannot be equal to 3 more than the digit sum of the prime, and the product cannot be a Smith number.

The same type of argument holds for multiplying 3^m, any power of 3, by a prime p other than 3. The product $3^m p$ is divisible by 3, but the sum $3^m + p$ is not. So, a prime multiplied by a power of 3 cannot be a Smith number.

Can a prime multiplied by a multiple of 3, other than a power of 3, be a Smith number?

Yes, of course; Professor Wilansky's brother-in-law's phone number is proof of that!

References

[1] A. Wilansky, Smith numbers, Two-Year Coll. Math. J., 13 (1982) 21.
[2] S. Oltikar and K. Wayland, Construction of Smith numbers, this MAGAZINE, 56 (1983) 36–37.
[3] H. Dubner, letter to Samuel Yates, Sept. 4, 1985.
[4] Samuel Yates, Sinkers of the titanics, J. Recreational Math., 17 (1984–85) 268–274.
[5] W. L. McDaniel, The existence of infinitely many k-Smith numbers, to be published in Fibonacci Quarterly.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by June 1, 1987

1252. *Proposed by Sydney Bulman-Fleming and Edward T. H. Wang, Wilfrid Laurier University, Canada.*

Let $L(n)$ denote the least common multiple of $\{1,2,\dots,n\}$. Color each positive integer red or blue according to the following rule: number 1 is red, and for $n>1$, n has the same color as $n-1$ if and only if $L(n)=L(n-1)$.

a. Find all instances of four consecutive integers with alternating colors.
b. Show that there are arbitrarily long sequences of consecutive integers with the same color.
c. Show that there are arbitrarily long sequences of consecutive red integers.

1253. *Proposed by Gerald A. Heuer, Concordia College, Minnesota.*

Suppose that x and y are related by $e^{-x}-e^{-y}=e^{-2}$. Prove that

$$\frac{1}{x}-\frac{1}{y}$$

is decreasing for x in the open interval $(0,2)$.

1254. *Proposed by Ambati Jaya Krishna (student), Johns Hopkins University, Maryland, and Mrs. Gomathi S. Rao, Orangeburg, South Carolina.*

Find the value of the continued fraction

$$1+\frac{2}{3+}\frac{4}{5+}\frac{6}{7+}\cdots.$$

1255. *Proposed by Harry D. Ruderman, Lehman College, The Bronx, New York.*

Prove that the number of odd coefficients in the expansion of

$$(x_1+x_2+\cdots+x_t)^n$$

is t^d, where d is the sum of the digits in the binary representation of n.

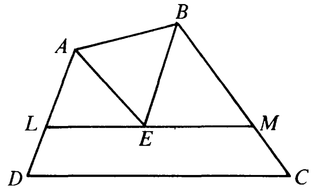
ASSISTANT EDITORS: Clifton Corzatt and Theodore Vessey, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1256. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Let $ABCD$ be a cyclic quadrilateral, let the angle bisectors at A and B meet at E, and let the line through E parallel to side CD intersect AD at L and BC at M. Prove that $LA + MB = LM$.



Quickies

Solutions to Quickies appear on p. 305.

Q715. *Proposed by David Callan, University of Bridgeport, Connecticut.*

Suppose $a_1, a_2, \ldots, a_n$ are real numbers. Obviously, if $a_1, a_2, \ldots, a_n$ are all positive, then

$$\sum_i a_i, \quad \sum_{i<j} a_i a_j, \quad \sum_{i<j<k} a_i a_j a_k, \ldots, a_1 a_2 \cdots a_n$$

are all positive. Prove that the converse is also true.

Q716. *Proposed by Norman Schaumberger, Bronx Community College.*

Show that for all positive integers n,

$$n^{\frac{1}{n}} + n^{\frac{1}{n+1}} + \cdots + n^{\frac{1}{2n-1}} \geqslant n\sqrt[n]{2}.$$

Solutions

Taylor Expansion **November 1985**

1226. *Proposed by L. Matthew Christophe, Jr., Wilmington, Delaware.*

Let

$$P(\lambda) = \int_0^\infty \frac{dx}{e^x + \lambda x}.$$

Find the Taylor expansion about the origin, and evaluate $P(1)$ and $P(-1)$ to ten decimal places.

Solution by G. A. Heuer, Concordia College, Minnesota.

If $f(x,\lambda)=(e^x+\lambda x)^{-1}$, then

$$\frac{\partial^n f}{\partial\lambda^n}=(-1)^n n!x^n(e^x+\lambda x)^{-n-1},$$

as one easily establishes by induction. To show that

$$P^{(n)}(\lambda)=\int_0^\infty\frac{\partial^n f}{\partial\lambda^n}\,dx$$

for each n and for $|\lambda|\le 2$ (say), it suffices to show that for each n the integral is uniformly convergent for $|\lambda|\le 2$. Now,

$$g(x,\lambda)=e^x+\lambda x\ge e^x-2x\ge 2(1-\ln 2)>0,$$

as one sees by examining g_x. Thus the integral in question converges uniformly by comparison with

$$\int_0^\infty n!x^n(e^x-2x)^{-n-1}\,dx.$$

Using the substitution $y=(n+1)x$ we obtain

$$P^{(n)}(0)=(-1)^n n!\int_0^\infty x^n e^{-(n+1)x}\,dx=(-1)^n a_n\int_0^\infty y^n e^{-y}\,dy=(-1)^n a_n n!,$$

where $a_n=n!(n+1)^{-(n+1)}$, so the Taylor series is

$$\sum_{n=0}^{\infty}(-1)^n a_n\lambda^n,$$

which converges for $|\lambda|<e$.

$P(1)$ lies between the sums $\sum_{n=0}^{22}(-1)^n a_n$ and $\sum_{n=0}^{23}(-1)^n a_n$, and this gives us $P(1)=.8063956162$ to 10 places. The series for $P(-1)$ has the ratio

$$\frac{a_{n+1}}{a_n}=\left(\frac{n+1}{n+2}\right)^{n+2}<e^{-1},$$

so that

$$\sum_{n=0}^{23}a_n<P(-1)<\sum_{n=0}^{23}a_n+a_{24}\left(\frac{e}{e-1}\right),$$

and $P(-1)=1.3590982771$ to 11 digits.

Also solved by Geoffrey A. Boyes (Canada), Paul Bracken, Duane Broline, M. L. Glasser, Russell Euler, Ismor Fischer, Zachary Franco, P. L. Hon (Hong Kong), Hans Kappus (Switzerland), Eusebio L. Koh (Canada), L. Kuipers (Switzerland) and M. Kuipers (Netherlands), Kee-wai Lau (Hong Kong), Irene Peters, Bjorn Poonen (student), Daniel M. Rosenblum (student), Volkhard Schindler (East Germany), Michael Vowe (Switzerland), Harry Weingarten, and the proposer.

Uniform Convergence of Series — November 1985

1227. *Proposed by G. A. Edgar, The Ohio State University.*

For real x let

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{1+n^2x^{2n}}.$$

Where is f continuous?

Solution by Tadeusz Figiel, The Ohio State University.

Let

$$f_n(x) = \frac{x^n}{1 + n^2 x^{2n}} \quad \text{for} \quad n = 1, 2, \dots .$$

We will show that the series $\Sigma_{n=0}^{\infty} f_n(x)$ converges uniformly on R and hence $f(x)$ is continuous on R. In fact, there is a constant $C < \infty$ such that for $N \geq 3$,

$$\sum_{n=N}^{\infty} |f_n(x)| \leq \frac{C}{\log N} \quad \text{for all } x \in R. \tag{1}$$

Because $|f_n(x)| \leq 1/n^2$ for all $x \geq 1$ and $|f_n(-x)| = |f_n(x)|$ for all $x \in R$, it clearly suffices to show that (1) holds for all $x \in (0,1)$.

Let $x \in (0,1)$ and $N \geq 3$ and let $m = \inf\{n \geq N\colon nx^n \leq 1\}$. Applying the Mean Value Theorem to the function $g(t) = m^t$ on the interval $[0, \frac{1}{m}]$ we have

$$m^{1/m} - 1 = m^{1/m} - m^0 = \frac{\log m}{m} m^c$$

for some $c \in (0,1)$, and, therefore,

$$m^{1/m} - 1 \geq \frac{\log m}{m}.$$

It follows that $m^{1+1/m} \geq m + \log m$ and thus, $m^{-1/m} \leq m/(m + \log m)$, so

$$1 - m^{-1/m} \geq \frac{\log m}{m + \log m}.$$

Since $x \leq m^{-1/m}$, we get

$$(1-x)^{-1} \leq (1 - m^{-1/m})^{-1} \leq \frac{m + \log m}{\log m}. \tag{2}$$

Thus,

$$\begin{aligned} S_1 = \sum_{n=m}^{\infty} f_n(x) &\leq \sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x} \\ &\leq \left(\frac{1}{m}\right)\left(\frac{m + \log m}{\log m}\right) \leq \frac{2}{\log m} \leq \frac{2}{\log N}. \end{aligned} \tag{3}$$

If $m = N$ we are done, so assume that $m > N$. Then

$$S_2 = \sum_{n=\left[\frac{m+1}{2}\right]}^{m-1} f_n(x) \leq \sum_{n=\left[\frac{m+1}{2}\right]}^{m-1} \frac{1}{n^2 x^n} \leq \left(\frac{2}{m}\right)^2 \sum_{n=1}^{m-1} x^{-n} = \frac{4}{m^2}\left(\frac{x^{1-m}}{1-x}\right).$$

Because $m > N$, the definition of m implies that $x^{m-1} > 1/(m-1)$. Therefore, using inequality (2) we get

$$S_2 \leq \frac{4}{m^2}(m-1)\frac{m + \log m}{\log m} \leq \frac{8}{\log m} \leq \frac{8}{\log N}. \tag{4}$$

Now we use the fact that $t^{-2}x^{-t}$ is an increasing function for $t \geq -\frac{2}{\log x}$. Letting $K = [-2/\log x]$, we have

$$S_3 = \sum_{n=K}^{\left[\frac{m+1}{2}\right]-1} f_n(x) \leq \frac{m-1}{2}\left(\frac{2}{m-1}\right)^2 x^{(1-m)/2}$$

$$\le \frac{2}{m-1}\sqrt{m-1}$$

$$= \frac{2}{\sqrt{m-1}} \le \frac{2}{\sqrt{N}} \le \frac{2}{\log N}. \tag{5}$$

Finally, if $N \le n < K$, then $x^{-n} < x^{-K} \le (x^{-1})^{-2/\log x} = e^2$. Hence,

$$S_4 = \sum_{n=N}^{K-1} f_n(x) \le e^2 \sum_{n=N}^{K} \frac{1}{n^2} \le e^2 \sum_{n=N}^{\infty} \frac{1}{n^2}$$

$$\le e^2 \frac{N-1}{N^2} \le \frac{2e^2 N}{N^2} = \frac{2e^2}{N} \le \frac{2e^2}{\log N}. \tag{6}$$

From (3), (4), (5), and (6), we conclude that

$$\sum_{n=N}^{\infty} f_n(x) = S_1 + S_2 + S_3 + S_4 \le \frac{C}{\log N},$$

where $C = 12 + 2e^2$.

Also solved by Irl C. Bivens, Bjorn Poonen (student), A. Meir (Canada). There were two incorrect solutions and one incomplete solution.

A Markov Chain Inequality — November 1985

1228. *Proposed by David Callan, University of Bridgeport.*

Let P be a stochastic matrix (all entries are nonnegative and each row sum is 1). Prove that for each positive integer n the largest entry in each column of

$$I + P + P^2 + \cdots + P^n$$

occurs on the diagonal.

I. *Solution by Jeff Benedict, Lanham, Maryland.*

Let $P = (p_{ij})$ be an $m \times m$ stochastic matrix and set

$$A(n) = (a_{ij}(n)) = I + P + P^2 + \cdots + P^n.$$

Since $A(n) = A(n-1) + P^n$ and all entries of P^n are nonnegative,

$$a_{ij}(n) \ge a_{ij}(n-1) \tag{1}$$

for $i, j = 1, 2, \dots, m$.

We will use induction to prove that

$$a_{ij}(n) \leqq a_{jj}(n-1), \qquad i \ne j, \tag{2}$$

for $i, j = 1, 2, \dots, m$, $i \ne j$.

For $i \ne j$, $a_{jj}(0) = 1 \ge p_{ij} = a_{ij}(1)$, so (2) is true for $n = 1$. Assume that (2) is true for n. Then, using the fact that $A(n+1) = I + PA(n)$, we have for $i \ne j$,

$$a_{ij}(n+1) = \sum_{k=1}^{m} p_{ik} a_{kj}(n). \tag{3}$$

That is, $a_{ij}(n+1)$ is a weighted average of $a_{1j}(n), \dots, a_{mj}(n)$. From the induction assumption, if $k \ne j$,

$$a_{kj}(n) \le a_{jj}(n-1) \le a_{jj}(n),$$

and, therefore, (3) implies that

$$a_{ij}(n+1) \le a_{jj}(n).$$

This completes the induction.

But this completes the proof since from (2) and (1),

$$a_{ij}(n) \le a_{jj}(n-1) \le a_{jj}(n) \quad \text{for } i, j = 1, 2, \ldots, m, \qquad i \ne j.$$

II. *Solution by Bjorn Poonen (student), Harvard College.*

We shall prove first that if P has no zero entries, then the entries along the diagonal of $S = I + P + \cdots + P^n$ are *strictly* greater than the other entries in their columns.

Let

$$P = (p_{ij}), \qquad S = (s_{ij}), \qquad T = (t_{ij}) = (P - I)S = P^{n+1} - I,$$

where $p_{ij} > 0$. Let the matrices be $m \times m$. Now suppose $s_{ij} \ge s_{kj}$ for all k. Then,

$$t_{ij} = \sum_{k=1}^{m} p_{ik} s_{kj} - s_{ij} \le \sum_{k=1}^{m} p_{ik} s_{ij} - s_{ij} = 0.$$

But P^{n+1} has only positive entries, so the only nonpositive entries of $T = P^{n+1} - I$ are along the diagonal. Thus, $i = j$, so the unique largest entry in each column of S is on the diagonal.

Now suppose P is any stochastic matrix (possibly with zero entries). Let A be any stochastic matrix without zero entries, and let $F(x) = xP + (1-x)A$. The condition that the largest entry in each column of $I + F(x) + \cdots + (F(x))^n$ occurs on the diagonal can be written as a system of polynomial inequalities in x. For $x \in [0,1)$, $F(x)$ is a stochastic matrix without zero entries, so these inequalities hold (without equality) by our previous result. But polynomials are continuous, so the inequalities hold (possibly with equality) at $x = 1$, when $F(x) = P$.

III. *Solution by Simon Tavare, University of Utah.*

Let $\{X_n : n = 0, 1, 2, \ldots\}$ be the Markov chain associated with the transition matrix P and state space S. For i, j in S and $X_0 = i$, define $T_{ij}(n)$ to be the number of times $X_m = j$ for $m = 0, 1, 2, \ldots, n$. Then

$$E(T_{ij}(n)) = \sum_{r=0}^{n} E(I\{X_r = j\} \mid X_0 = i) = \left(\sum_{r=0}^{n} P^r\right)_{ij}. \tag{1}$$

(Here, $I\{X_r = j\}$ is an indicator random variable which equals 1 or 0 depending upon whether or not $X_r = j$.)

For $X_0 = i$ and $i \ne j$, let $V_{ij} = \min\{n > 0 : X_n = j\}$. If $V_{ij} > n$, then $T_{ij}(n) = 0$, whereas if $1 \le m \le n$ and $V_{ij} = m$, then $T_{ij}(n) = T^*_{jj}(n-m)$ (note that $T_{jj}(0) = 1$), where for each k, $T^*_{jj}(k)$ has the same distribution as $T_{jj}(k)$. It follows that

$$\begin{aligned} E(T_{ij}(n)) &= \sum_{m=1}^{n} \Pr(V_{ij} = m) E(T_{jj}(n-m)) \\ &\le \sum_{m=1}^{n} \Pr(V_{ij} = m) E(T_{jj}(n)) \le E(T_{jj}(n)). \end{aligned}$$

Combining the last inequality with (1) gives the desired result. Note that this argument holds even when P is a countably infinite matrix.

Also solved by P. Banerjee and B. Viswanathan (Canada), Duane Broline, Martial Herbert, Western Maryland College Problem Group, and the proposer.

Linear Dependence of Coefficients November 1985

1229. *Proposed by Roger L. Creech, East Carolina University.*

Let $P(x)$ be a polynomial of degree $n > 0$ with coefficients in Q, the field of rational numbers. Let α be any complex number. Then $P(x) = (x - \alpha)q(x) + r$, where $q(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0$ and r is a complex number. Prove that the set

$$S = \{ c_{n-1}, c_{n-2}, \dots, c_0, r \}$$

is linearly dependent over Q if and only if α is a zero of a polynomial of degree n in $Q[x]$.

Solution by Dennis Hamlin (student), University of Minnesota.

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$, with $a_n, \dots, a_0 \in Q$. Equating coefficients in $P(x) = (x - \alpha)q(x) + r$ gives

$$a_n = c_{n-1}, \qquad a_{n-1} = c_{n-2} - \alpha c_{n-1}, \dots, \qquad a_1 = c_0 - \alpha c_1, \qquad a_0 = r - \alpha c_0.$$

Thus,

$$\begin{aligned} c_{n-1} &= a_n, \\ c_{n-2} &= a_{n-1} + \alpha c_{n-1} = a_{n-1} + \alpha a_n, \\ c_{n-3} &= a_{n-2} + \alpha c_{n-2} = a_{n-2} + \alpha a_{n-1} + \alpha^2 a_n, \\ &\vdots \\ c_0 &= a_1 + \alpha c_1 = a_1 + \alpha a_2 + \cdots + \alpha^{n-1} a_n, \\ r &= a_0 + \alpha c_0 = a_0 + \alpha a_1 + \cdots + \alpha^n a_n = P(\alpha). \end{aligned}$$

Thus $S \subseteq Q[\alpha]$. If α satisfies a polynomial of degree n over Q, then $\dim_Q Q[\alpha] \le n$, so the $n+1$ elements in S must be linearly dependent. Conversely, if $d_0, \dots, d_n \in Q$, not all zero, give $d_0 c_0 + \cdots + d_{n-1} c_{n-1} + d_n r = 0$, then α satisfies the polynomial

$$A(x) = d_0(a_1 + a_2 x + \cdots + a_n x^{n-1}) + \cdots + d_{n-2}(a_{n-1} + a_n x) + d_{n-1} a_n + d_n P(x),$$

which is in $Q[x]$ and has degree $\le n$.

Also solved by Duane Broline, Zachary Franco, Enzo R. Gentile (Argentina), Robert Gilmer and Budh Nashier, Kee-wai Lau (Hong Kong), Mark Leeney, Charles de Matas (West Indies), Bjorn Poonen (student), Dennis Spellman, Gary L. Walls, William P. Wardlaw, Yan-loi Wong (student), and the proposer.

Similar Triangles November 1985

1230. *Proposed by L. Kuipers, Sierre, Switzerland.*

Let ABC and $A'B'C'$ be two similar and similarly oriented triangles in a plane. Let $AA'A''$, $BB'B''$, and $CC'C''$ be three triangles lying in the plane and similar and similarly oriented to ABC. Prove that triangle $A''B''C''$ is similar and similarly oriented to ABC.

I. *Solution by W. Weston Meyer, General Motors Research Laboratories.*

We will treat the plane as complex and the point labels A, B, C, etc., as complex numbers when used algebraically. This allows us to characterize all triangles PQR that are similar and similarly oriented to a given triangle ABC in the following way:

$$P(B - C) + Q(C - A) + R(A - B) = 0. \tag{1}$$

For the equation is satisfied with $(P, Q, R) = (A, B, C)$ and remains so under the transformation

$$(P, Q, R) \to (PK + H,\ QK + H,\ RK + H)$$

representing arbitrary rotation and radial scale-change K, followed by arbitrary translation H. Paraphrasing (1), we may brand (P, Q, R) *orthogonal* to $(L, M, N) = (B - C, C - A, A - B)$.

Absent degeneracy, the components of (L, M, N) are nonzero.

In the problem at hand, the common similarity of $AA'A''$, $BB'B''$, and $CC'C''$ to ABC is expressible

$$\begin{pmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{pmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

Since $A'B'C'$ is also similar, the first two column vectors of the 3×3 matrix are orthogonal to (L, M, N). Premultiplication by (L, M, N) on both sides of (2) yields

$$(LA'' + MB'' + NC'')N = 0,$$

showing the third column to be orthogonal too. This proves the proposition.

II. *Solution by Erzo R. Gentile, Ciudad Universitaria, Buenos Aires, Argentina.*

We work in the Gaussian plane and use the fact that the triangle with vertices z_1, z_2, z_3 is similar and similarly oriented to the triangle with vertices w_1, w_2, w_3 if and only if

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

The hypotheses are that

$$\frac{B - A}{C - A} = \frac{A - A'}{A - A''} = \frac{B' - B}{B'' - B} = \frac{C' - C}{C'' - C} = \frac{A' - B'}{A' - C'}.$$

From the first, second, and fourth of these we find that

$$\frac{B - A}{C - A} = \frac{B - A' + C' - C}{C'' - A''},$$

and from all of them we find that

$$\frac{C' - C}{B'' - A'' + C'' - B + A' - C'} = \frac{C' - C}{C'' - C},$$

provided that the denominators in each of these are nonzero.

The last equation implies that $B'' - A'' = B - A' + C' - C$, and substituting this into the previous equation we have

$$\frac{B - A}{C - A} = \frac{B'' - A''}{C'' - A''}.$$

This completes the proof.

Also solved by Jordi Dou (Spain), Howard Eves, Leonard D. Goldstone, Dr. Hans Kappus (Switzerland), M. S. Klamkin (Canada), R. C. Lyness (England), Bjorn Poonen (student), Raymond E. Rogers, Robert L. Young, and the proposer. There was one incomplete solution.

Several noted that the problem appears in the literature, for example, as Theorem 4.83, *Geometry Revisited*, H. S. M. Coxeter and S. L. Greitzer, New Mathematical Library. Kappus and Rogers noted that it may happen that $A'' = B'' = C''$, in which case $A''B''C''$ is degenerate.

Answers

Solutions to the Quickies which appear on p. 298

A715. Consider the polynomial

$$p(x) = \prod_i (x + a_i) = x^n + \sum_i a_i x^{n-1} + \sum_{i<j} a_i a_j x^{n-2} + \cdots + a_1 a_2 \cdots a_n.$$

By hypothesis, the coefficients are positive, so none of the roots is positive or zero. Thus, the roots, $-a_1, -a_2, \ldots, -a_n$, are all negative, and the result follows.

A716. Using the arithmetic-mean–geometric-mean inequality twice gives

$$n^{\frac{1}{n}} + n^{\frac{1}{n+1}} + \cdots + n^{\frac{1}{2n-1}} = \frac{n^{\frac{n+1}{n}} + n^{\frac{n+2}{n+1}} + \cdots + n^{\frac{2n}{2n-1}}}{n}$$

$$\geqslant \left(n^{\frac{n+1}{n}} \cdot n^{\frac{n+2}{n+1}} \cdots n^{\frac{2n}{2n-1}} \right)^{\frac{1}{n}} = n^{\left(\frac{n+1}{n} + \frac{n+2}{n+1} + \cdots + \frac{2n}{2n-1} \right)\frac{1}{n}}$$

$$\geqslant n^{\left(\frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdots \frac{2n}{2n-1} \right)^{\frac{1}{n}}} = n^{\sqrt[n]{2}}.$$

REVIEWS

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

History Project (Box 3716, Santa Rosa, CA 95402), 1980; viii + 257 pp, $12.95. *Teacher's Manual*, by Teri Hoch Perl and Patsy Christner, 21 pp.

This book was written to provide role models (primarily for elementary- and middle-school students) to encourage girls to aspire to careers in math-related fields. Included are biographical sketches of both historic and contemporary women, who worked in a variety of careers requiring substantial training in mathematics. Is this book for your students? More likely, it's for your children, or your younger sisters--and for their teachers. Put it in their hands.

Rouvray, Dennis H., Predicting chemistry from topology, *Scientific American* 255:3 (September 1986) 40-47, 120.

"Methods relying on the topology of molecules ... but ignoring their three-dimensional shapes predict a broad range of properties." In particular, properties are commonly deduced from molecular connectivity indexes such as the carbon number (= number of vertices in the graph of the molecule), the Wiener index (= sum over all pairs of vertices of the minimum number of edges from one vertex to the other), and the Randic index (= sum over all edges of $1/\sqrt{d_1 d_2}$, where d_1 and d_2 are the degrees of the vertices the edge joins).

Among the many properties correlated with these indexes are the tendencies of molecules to act as anesthetics, narcotics, and hallucinogens.

Rourke, Colin, and Ian Stewart, Poincaré's perplexing problem, *New Scientist* (4 September 1986) 41-45.

Perhaps British mathematicians have the right idea: if you make a big discovery, why not announce it in the country's major popular science magazine in an article you write yourself, rather than wait for months for the details to be published in a mathematical journal, and even longer for some science writer to call (who will likely get some of the details wrong anyhow)? Undaunted by its premature announcement two years ago that the Riemann Hypothesis has been proved, *New Scientist* has published a claim that the final case of the Poincaré conjecture has been proved by C. Rourke (Warwick) and E. Rego (Oporto). That case, for n = 3, states that the only simply-connected 3-manifold is the 3-sphere. Michael Freedman, the recent Fields Medalist, disposed of the case n = 4 in 1982. Rourke, one of the authors of the article, does caution that "the problem is notorious for its subtlety and no one can yet be absolutely certain that it is solved." The article relates the history of the conjecture and gives some hints about the methods involved in resolving the final case.

Peterson, Ivars, Untangling a knotty problem: mathematicians find a new, simple way to distinguish different types of knots, *Science News* 128 (1985) 266.

Simultaneous discovery strikes again--this time, five groups of mathematicians (with three different approaches) found a new polynomial function that distinguishes knots. Still elusive: a complete invariant, so that different knots would have different polynomials.

Colin, Stewart T., *Puzzle Craft*, 1985 edition, (79 Old Sudbury Road, Lincoln, MA 01773); ii + 100 pp, $12 (P).

Compendium of information on mechanical and geometrical puzzles, "mostly about puzzles you can design and make," by the premier wood-puzzle maker in the U.S. Also included are hints on woodworking techniques, puzzle-making as a business, and a catalogue of all wooden puzzles made by the author since 1968.

Korf, Richard E., *Learning to Solve Problems by Searching for Macro-Operators*, Pitman, 1985; v + 147 pp (P).

Algorithms for Rubik's cube cannot be arrived at by the current stock of "weak methods" (general problem-solving strategies such as heuristic search, guess-and-test, hill-climbing, and means-ends analysis), because later steps in the solution demand violating previously-satisfied subgoals. The key is to keep the violations temporary and restore the previous subgoals at the end of each "macro" of moves. The author shows how his Macro Problem Solver generates efficient algorithms for the cube, the Fifteen Puzzle, Think-a-Dot, and the Towers of Hanoi. This is the rare doctoral thesis that is valuable reading for the general mathematical public.

Crypton, Dr., Perils of the football draft; two game-theorists find a flaw in the pro-football draft system, *Science Digest* (July 1986) 76-79.

Very entertaining rendition of the paradoxes of the football draft discovered by P. D. Straffin, Jr. (Beloit) and S. J. Brams (NYU) and related in their 1978 article in the *American Mathematical Monthly*. The situation of the draft is one of Prisoners' Dilemma, in which the team "allowed to pick first may beg for the opportunity to pick last--and the other teams, looking out for their own interests, will support this request."

Sharenow, Ira, Doubling at backgammon, *ORSA Student Communications* 3:1 (Fall 1984) 5-7; Doubling at backgammon II, 3:2 (Winter 1985) 4-7.

Summarizes known theory of backgammon doubling, in the very readable form of a conversation between a professor and two students.

Denning, Peter J., The science of computing: will machines ever think?, *American Scientist* 74:4 (July-August 1986) 344-346.

Denning's continuing column tackles the key question of artificial intelligence, summarizing perspectives of recent writers. Conclusion: "Perhaps stored-program computers can't really think, but maybe another kind of computer ..."

Peterson, Ivars, Inside averages, *Science News* 129 (10 May 1986) 300-302.

Describes work of R. L. Graham (Bell Labs) and P. Diaconis (Stanford) on reconstructing a function from various projections or averages. In such "projection pursuit" methods, a simple version of the Radon transform idea is used to look for patterns in data. Diaconis used these methods to determine the order in which Plato wrote his books.

Peterson, Ivars, Games mathematicians play, *Science News* 130:12 (20 September 1986) 186-189.

Dwells on the glories and satisfactions of recreational mathematics.

Morgan, Frank, Soap films and problems without unique solutions, *American Scientist* 74:3 (May-June 1986) 232-236.

Replete with figures, this article investigates the question: Can a smooth boundary support infinitely many surfaces of least energy (least area)? Research of the past few years, including work of the author, shows that for $n < 8$, the answer is no in ordinary n-dimensional Euclidean space or in an n-dimensional manifold satisfying a few mild conditions. For $n = 8$ and above, the question is still open; and 7-dimensional soap films in 8-space may have "troublesome corners."

Tritton, David, Chaos in the swing of a pendulum, *New Scientist* (24 July 1986) 37-40; Letters: Chaotic pendulum, (7 August 1986) 57.

A ball on the end of a string can provide an extremely simple example of mathematically chaotic behavior: consecutive experiments under apparently identical conditions produce different patterns of motion. (The point of suspension must be forced to oscillate in a straight line at a frequency close to the natural frequency of the pendulum; the ball develops a motion also in the direction perpendicular to the drive.) So, curiously, "determinism in principle does not contradict the lack of predictability in practice." The key to the conundrum is sensitivity to initial conditions; for chaotic systems, the smallest change leads ultimately to very different results. "Deterministic chaos has profound implications both for our understanding of nature and for our attempts to forecast the behaviour of systems of practical importance."

Peterson, Ivars, Keeping secrets: how to prove a theorem so that no one else can claim it, *Science News* 130 (30 August 1986) 140-141.

In one of the letters he sent to Leibniz through Oldenburg, Newton closed with a cryptogram that described the basis of his methods. Now mathematicians have devised even more clever schemes for a person to demonstrate knowledge of a proof of a result without revealing the proof itself or anything about it (except an upper bound on its length).

Stewart, Ian, Mathematics: Demystifying the monster, *Nature* 319 (20 February 1986) 621-622.

The "monster" is the largest sporadic simple group, "containing a septemdecillion or so elements," and it is the group of symmetries of an algebra of 196,884 dimensions. J. H. Conway (Cambridge) has given a simpler construction of the monster, one that "involves a whole horde of exceptional combinatorial objects": Golay codes, Steiner triples, the Leech lattice (related to sphere-packing in 24-space), and more. "Somehow these exotic creatures ... are all part of the same big picture, whatever that may be. This assortment of ingredients is combined in the mathematical equivalent of a witches' cauldron to produce a powerful potion, an algebra of 196,884 dimensions which is '99 per cent associative' ... . Is the time ripe for a theory of 'almost laws' of algebra?"

Kolata, Gina, Prime tests and keeping proofs secret, *Science* 233 (29 August 1986) 938-939.

L. Adleman and M. D. Huang (USC) have developed a new method of testing for primes that is based on the 1983 result that won G. Faltings a Fields Medal this past August. The mathematics they used involves abelian varieties over finite fields; their prime-testing method is probabilistic. In a separate development, other mathematicians have shown that it is possible to convince a person that a theorem is true without providing any details of a proof; M. Blum (Berkeley) has shown how to do such a "zero-knowledge" proof for any mathematical theorem.

NEWS & LETTERS

1986 USA AND CANADIAN OLYMPIADS: SOLUTIONS

The solutions which follow have been especially prepared for publication in the MAGAZINE by Loren Larson and Bruce Hanson of St. Olaf College.

15th USA MATH OLYMPIAD

1. Part a. Do there exist 14 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \le p \le 11$?

Part b. Do there exist 21 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \le p \le 13$?

Sol. The answers are (a) No, and (b) Yes.

a. Let the 14 consecutive numbers be $N, N+1, N+2, \ldots, N+13$. By symmetry, it does not matter whether N is even or odd, so suppose N is even. Let S denote the set of elements in the sequence $N, N+1, N+2, \ldots, N+13$ that are not divisible by either 2 or 3. The elements in S depend upon the congruence class of N modulo 3, as seen in the following table.

N(mod 3)	S
(i) 0	$N+1$, $N+5$, $N+7$, $N+11$, $N+13$
(ii) 1	$N+1$, $N+3$, $N+7$, $N+9$, $N+13$
(iii) 2	$N+3$, $N+5$, $N+9$, $N+11$

In cases (i) and (ii), S contains five elements. At most two of these are divisible by 5, at most one is divisible by 7, and at most one is divisible by 11. This leaves at least one that is not divisible by a prime p, $2 \le p \le 11$.

In case (iii), S contains four elements. At most one of these is divisible by 5, at most one by 7, and at most one by 11. This leaves at least one that is not divisible by a prime p, $2 \le p \le 11$.

b. If $N \equiv 0 \pmod 2$ then $N, N+2, N+4, \ldots, N+20$ are even. If $N \equiv 2 \pmod 3$ then $N+1, N+7, N+13, N+19$ are divisible by 3. If $N \equiv 0 \pmod 5$ then $N+5$, $N+15$ are divisible by 5. If $N \equiv 4 \pmod 7$, $N \equiv 2 \pmod{11}$, $N \equiv 2 \pmod{13}$ then $N+3$, $N+17$ are divisible by 7, $N+9$ and $N+11$ are divisible by 11 and 13 respectively. The existence of such an N is a consequence of the Chinese Remainder Theorem. (Alternatively, one can compute the smallest positive integer N with these properties, and one finds $N = 9{,}440$.) For such an integer, $N, N+1, N+2, \ldots, N+20$ are 21 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \le p \le 13$.

2. During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.

Sol. Construct a graph G with 10 vertices representing the naps taken by the mathematicians. Denote by $N(x)$ the nap represented by the vertex x. Two vertices x and y are joined by an edge if and only if the naps $N(x)$ and $N(y)$ overlapped.

Suppose G has k connected components with $n_1, n_2, \ldots, n_k$ vertices respectively. Suppose that none of these components contains a cycle (i.e., a closed path formed by three or more edges). Then each component is a tree, so the number of edges in G is $(n_1-1) + (n_2-1) + \ldots + (n_k-1) < n_1 + n_2 + \ldots + n_k = 10$. But G has at least 10 edges since every two mathematicians had overlapping naps. This contradiction implies that G contains a cycle C.

Let x be the vertex on C for which the nap $N(x)$ was the first to end. Let y and z be the vertices which are adjacent to x along C. Then, just as the nap $N(x)$ was concluding, the naps $N(y)$ and $N(z)$ were still in progress. It follows that three mathematicians were sleeping at that time.

3. What is the smallest integer n, greater than one, for which the root-mean-square of the first n positive integers is an integer?

Note. The root-mean-square of numbers $a_1, \ldots, a_n$ is defined to be $[(a_1^2 + \ldots + a_n^2)/n]^{1/2}$.

Sol. Suppose that $(1^2 + 2^2 + \ldots + n^2)/n = n(n+1)(2n+1)/6n = (n+1)(2n+1)/6$ is a perfect square m^2.

We first observe that $(n+1)(2n+1) \equiv 0 \pmod 6$ if and only if $n \equiv 1,5 \pmod 6$. We consider each case.

Suppose that $n = 6k+5$. Then $m^2 = (k+1)(12k+11)$. But $k+1$ and $12k+11$ are relatively prime, so both are perfect squares. Let $k+1 = s^2$ and $12k+11 = t^2$. From these we find that $12s^2 = t^2+1$. But this is impossible because $12s^2 \equiv 0 \pmod 4$ whereas $t^2+1 \equiv 1$ or $2 \pmod 4$.

Suppose that $n = 6k+1$. Then $m^2 = (3k+1)(4k+1)$. Again, $3k+1$ and $4k+1$ are relatively prime, so both are perfect squares. Consider the sequence of squares. Those of the form $4k+1$ correspond to k = 0, 2, 6, 12, 20, 30, 42, 56, ... and those of the form $3k+1$ correspond to k = 0, 1, 5, 8, 16, 21, 33, 40, 56, 65, ... If $k = 0$, we get $n = 1$ (the trivial case). The next value common to these sequences is $k = 56$, and this leads to the smallest nontrivial solution: $n = 6 \times 56 + 1 = 337$.

4. Two distinct circles K_1 and K_2 are drawn in the plane. They intersect at points A and B, where AB is a diameter of K_1. A point P on K_2 and inside of K_1 is also given.

Using only a "T-square" (i.e., an instrument which can product the straight line joining two points and the perpendicular to a line through a point on or off the line), find an explicit construction for:

two points C and D on K_1 such that CD is perpendicular to AB and CPD is a right angle.

Sol. We first construct the mirror image P' of P in AB. To do this, draw the line AP, which intersects the circle K_1 in A'. Draw perpendicular lines ℓ and ℓ' to AB from P and A' respectively. The line ℓ' intersects K_1 again at a point A''. Draw the line A''. Then A'' intersects ℓ at P'.

Now draw a line through P' perpendicular to ℓ. This line meets K_2 in points F and G. We can use either F or G; take F. Draw the line FP, which meets the diameter AB in a point E. The perpendicular to AB at E gives the desired chord CD.

To see this, observe that in K_2, $FE \cdot EP = AE \cdot EB$. Also, in K_1, $AE \cdot EB = CE \cdot ED$. Thus, $FE \cdot EP = CE \cdot ED$. But $FE = EP$ and $CE = ED$, so that $CE = EP = ED$. Thus, E is the center of a circle passing through C, P, and D, and CPD is a right angle.

5. By a *partition* π of an integer $n \geq 1$, we mean a representation of n as a sum of one or more positive integers, where the summands must be put in nondecreasing order. (E.g., if $n = 4$, then the partitions π are $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3$, $2 + 2$, and 4.)

For any partition π, define $A(\pi)$ to be the number of 1's which appear in π, and define $B(\pi)$ to be the number of distinct integers which appear in π. (E.g., if $n = 13$ and π is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$.)

Prove that, for any fixed n, the sum of $A(\pi)$ over all partitions π of n is equal to the sum of $B(\pi)$ over all partitions π of n.

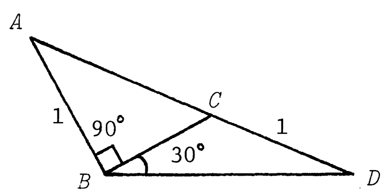
Sol. For a positive integer k, let $P_k(x) = 1 + x^k + x^{k+k} + x^{k+k+k} + \ldots$ and let $Q(x) = x + 2x^{1+1} + 3x^{1+1+1} + \ldots$. Let $\Pi = Q(x) \prod_{k=2}^{\infty} P_k(x)$. The coefficient of x^n in π is the sum of $A(\pi)$ over all partitions π of n.

Now $Q(x) = xP_1(x) + x^2P_1(x) + x^3P_1(x) + \ldots = (x + x^2 + x^3 + \ldots)P_1(x)$, so that $\Pi = (x + x^2 + x^3 + \ldots) \prod_{k=1}^{\infty} P_k(x)$.

Let $\prod_{k=1}^{\infty} P_k(x) = c_0 + c_1x + c_2x^2 + \ldots$. For $1 \leq k \leq n$, c_{n-k} is the number of partitions of n which contain the integer k. Thus, the number of partitions π of n in which k occurs, summed for $k = 1,2,3,\ldots,n$ is $c_{n-1} + c_{n-2} + \ldots + c_0$. This is the coefficient of x^n in $(x + x^2 + x^3 + \ldots) \prod_{k=1}^{\infty} P_k(x) = \Pi$. But observe that the number of partitions π of n in which k occurs, summed for $k = 1,2,\ldots,n$ is precisely the sum of $B(\pi)$ over all partitions π of n. This completes the proof.

18th CANADIAN MATH OLYMPIAD

1.



In the diagram AB and CD are of length 1 while angles ABC and CBD are 90° and 30° respectively. Find AC.

Sol. Let $x = BD$ and $y = AC$. By the law of sines, $\frac{x}{\sin BCD} = \frac{1}{\sin 30°} = 2$ and $\frac{y}{\sin 90°} = \frac{1}{\sin(180° - BCD)} = \frac{1}{\sin BCD}$. From these we find that $y = 2/x$.

By the law of cosines applied to triangle ABD, $(1+y)^2 = 1 + x^2 + x$. Substitution of $y = 2/x$ from above yields

$$(1 + 2/x)^2 = 1 + x + x^2,$$

$$(x+1)(x^3 - 4) = 0.$$

The only positive real root of this equation is $x = \sqrt[3]{4}$, which in turn gives $y = AC = \sqrt[3]{2}$.

2. A Mathlon is a competition in which there are M athletic events. Such a competition was held in which only A, B and C participated. In each event p_1 points were awarded for first place, p_2 for second place and p_3 for third where $p_1 > p_2 > p_3 > 0$ and p_1, p_2, p_3 are integers. The final score for A was 22, for B was 9 and for C was also 9. B won the 100 metres. What is the value of M and who was second in the high jump?

Sol. From $(p_1 + p_2 + p_3)M = 40$, $p_1 + p_2 + p_3 \geq 6$, and $M \geq 2$, we deduce that $M = 2$, 4, or 5.

Suppose $M = 2$. Then $p_1 \leq 8$ (consider B's score). This implies that A's score ≤ 16, contrary to hypothesis.

Suppose $M = 4$. Then $p_1 + 1 + 1 + 1 \leq B$'s score $= 9$, so $p_1 \leq 6$. If $p_1 \leq 5$, then A's score $\leq p_2 + 3\text{X}5 \leq 20$, a contradiction. Therefore $p_1 = 6$ and $p_3 = 1$ (consider B's score). Also A's score $p_2 + 3p_1 = p_2 + 18$, so that $p_2 \geq 4$. This implies that C's score $\geq p_3 + 3p_2 \geq 1 + 3\text{X}4 = 13$, a contradiction.

Therefore $M = 5$, and it easily follows that $p_1 = 5$, $p_2 = 2$, $p_3 = 1$, and that C placed second in the high jump.

3. A chord ST of constant length slides around a semicircle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

Sol. Extend SP to cut the circle again at U. ΔSPM is similar to ΔSUT and therefore $\angle SPM = \angle SUT$. But SUT is constant for all positions of ST.

4. For positive integers n and k, define $F(n,k) = \sum_{r=1}^{n} r^{2k-1}$. Prove that $F(n,1)$ divides $F(n,k)$.

Sol. The terms of $F(n,k)$ can be added "forwards and backwards" to give

$$2F(n,k) = \sum_{r=1}^{n} [r^{2k-1} + (n+1-r)^{2k-1}] .$$

Each term on the right is divisible by $n+1$.

We can also write the left side in the form

$$2F(n,k) = 2n^{2k-1} + \sum_{r=1}^{n-1} [r^{2k-1} + (n-r)^{2k-1}] .$$

In this case, each term on the right is divisible by n. Since n and $n+1$ are relatively prime, we see that $2F(n,k)$ is divisible by $n(n+1)$. It follows that $F(n,k)$ is divisible by $n(n+1)/2 = F(n,1)$.

5. Let u_1, u_2, u_3, ... be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$. Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

Sol. Observe that $u_3 = 1986$.

For an integer x, let $\overline{x}$ denote the unique integer, $0 \leq x < 1986$ such that $\overline{x} \equiv x \pmod{1986}$. Consider the sequence $\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots$. Because the $\overline{u}_i$'s are from a finite set, the sequence of ordered pairs $(\overline{u}_1,\overline{u}_2)$, $(\overline{u}_2,\overline{u}_3)$, $(\overline{u}_3,\overline{u}_4)$, ... must eventually repeat. Let $(\overline{u}_k,\overline{u}_{k+1})$ be the first ordered pair which is a repetition of an earlier pair, say $(\overline{u}_i,\overline{u}_{i+1})$, for some i, $1 \leq i < k$.

We claim that $i = 1$. For suppose not. From $\overline{u}_{i+1} = \overline{u}_{k+1}$ we have $u_{i+1} \equiv u_{k+1} \pmod{1986}$, and therefore $u_i^2 - u_{i-1} \equiv u_k^2 - u_{k-1} \pmod{1986}$. In a similar manner, $u_i \equiv u_k \pmod{1986}$, and therefore the previous equation implies that $u_{i-1} \equiv u_{k-1} \pmod{1986}$. But then $(\overline{u}_{k-1},\overline{u}_k)$ is a repeat of $(\overline{u}_{i-1},\overline{u}_i)$, contrary to our choice of k.

Thus, $\overline{u}_k = \overline{u}_1$, $\overline{u}_{k+1} = \overline{u}_2$ and therefore $\overline{u}_1,\overline{u}_2,\overline{u}_3$, ... is a periodic sequence with cycle length k. It follows that $\overline{u}_{jk+3} = 0$ for each positive integer j, or equivalently, u_{jk+3} is divisible by 1986 for each positive integer j.

27th INTERNATIONAL MATH OLYMPIAD PROBLEMS

1. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a,b in the set $\{2,5,13,d\}$ such that $ab-1$ is not a perfect square.

2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a sequence of points P_1, P_2, P_3, ... such that P_{k+1} is the image of P_k under rotation with center A_{k+1} through angle 120° clockwise (for $k = 0,1,2,\ldots$). Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all the five numbers is positive. If three consecutive vertices are assigned the numbers x,y,z respectively and $y < 0$ then the following operation is allowed: the numbers x,y,z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure *necessarily* comes to an end after a finite number of steps.

4. Let A,B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that

(i) $f[xf(y)]f(y) = f(x+y)$ for all $x,y \geq 0$,

(ii) $f(2) = 0$,

(iii) $f(x) \neq 0$ for $0 \leq x < 2$.

6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white points and red points on L is not greater than 1? Justify your answer.

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Feroe, John, *Vassar College*
Firey, William J., *Oregon State Univ.*
Fishburn, Peter C., *AT&T Bell Labs, Murray Hill, New Jersey*
Fisher, Evan, *Lafayette College*
Fisher, J. Chris, *University of Regina*
Foster, Lorraine L., *California State University, Northridge*
Fraser, Craig, *Victoria College*
Fraser, Marshall, *San Francisco, Calif.*
Gilbert, William J., *University of Waterloo*
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Goldstein, Jerome A., *Tulane Univ.*
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Green, Euline "Bo", *Abilene Christian University*
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Guy, Richard, *University of Calgary*
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Harborth, Heiko, *Braunschweig, West Germany*
Harrell, Ronald E., *Allegheny College*
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Howard, F.T., *Wake Forest University*
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Jacobson, Bernard, *Franklin & Marshall College*
James, David, *University of Michigan--Dearborn*
Jamison, Robert E., *Clemson University*
Johnson, Marjorie, *Santa Clara, Calif.*
Johnsonbaugh, Richard F., *DePaul Univ.*
Kalman, Dan, *Rancho Palos Verdes, CA*
Katz, Victor J., *Silver Spring, MD*
Keller, Edward L., *California State University, Hayward*

Klamkin, Murray S., *Univ. of Alberta*
Klosinski, Leonard, *Santa Clara Univ.*
Konhauser, Joseph D.E., *Macalester Col.*
Kuzmanovich, James, *Wake Forest Univ.*
Laatsch, Richard G., *Miami University*
Lagarias, Jeffrey C., *AT&T Bell Labs., Murray Hill, New Jersey*
Lautzenheiser, Roger G., *Rose-Hulman Institute of Technology*
Levine, Alan, *Franklin & Marshall Col.*
L'heureux, James E., *West Chester Univ.*
Liu, A., *University of Alberta*
LoBello, Anthony, *Allegheny College*
Logothetti, Dave, *Santa Clara Univ.*
Loud, Warren S., *Univ. of Minnesota*
Marcus, Marvin, *University of California, Santa Barbara*
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